Signals and Systems

Fall 2003

Lecture #9

2 October 2003

1. The Convolution Property of the CTFT
2. Frequency Response and LTI Systems Revisited
3. Multiplication Property and
   Parseval's Relation
4. The DT Fourier Transform

The CT Fourier Transform Pair

$$x(t) \longleftrightarrow X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \qquad \text{– FT}$$

(Analysis Equation)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \qquad \text{– Inverse FT}$$

(Synthesis Equation)

Last lecture: some properties
Today: further exploration

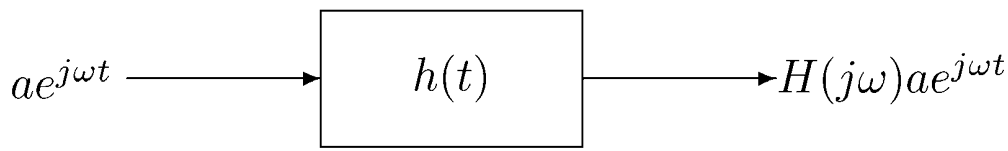
Convolution Property

$$y(t) = h(t) * x(t) \longleftrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

$$\text{where } h(t) \longleftrightarrow H(j\omega)$$

A consequence of the eigenfunction property:

$$x(t) = \int_{-\infty}^{\infty} \underbrace{\left(\frac{1}{2\pi} X(j\omega) d\omega\right)}_{\text{coefficient } a} e^{j\omega t}$$

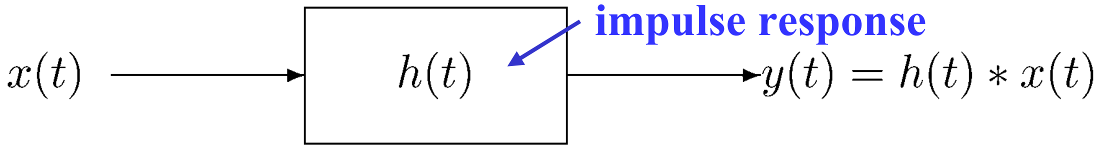


$$y(t) \quad = \quad \int_{-\infty}^{\infty} \underbrace{\left(H(j\omega)\frac{1}{2\pi} X(j\omega) d\omega\right)}_{H(j\omega)\cdot a} e^{j\omega t}$$

$$= \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{H(j\omega)X(j\omega)}_{Y(j\omega)} e^{j\omega t} d\omega$$

Synthesis equation for *y(t)*

The Frequency Response Revisited

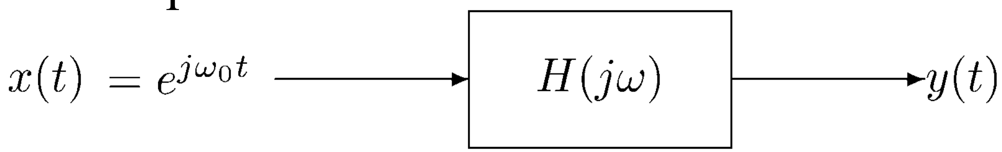


$$Y(j\omega) = H(j\omega)X(j\omega)$$

frequency response

The frequency response of a CT LTI system is simply the Fourier transform of its impulse response

Example #1:



Recall

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - \omega_0)$$

$$Y(j\omega) = H(j\omega)X(j\omega) = H(j\omega)2\pi\delta(\omega - \omega_0) = 2\pi H(j\omega_0)\delta(\omega - \omega_0)$$

$$\Downarrow \text{ inverse FT}$$

$$y(t) = H(j\omega_0)e^{j\omega_0 t}$$

Example #2: A differentiator

$$y(t) = \frac{dx(t)}{dt}$$ **- an LTI system**

Differentiation property: $Y(j\omega) = j\omega X(j\omega)$

$$\Downarrow$$

$$H(j\omega) = j\omega$$

1) Amplifies high frequencies (enhances sharp edges)

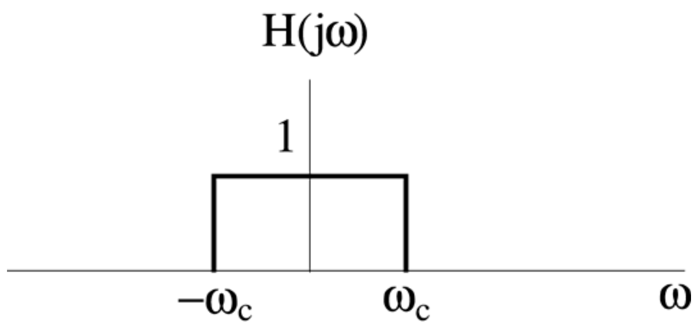
2) $+\pi/2$ phase shift $(j = e^{j\pi/2})$

Larger at high ω_o — phase shift

$$\frac{d}{dt}\sin\omega_0 t = \omega_0\cos\omega_0 t = \omega_0\sin(\omega_0 t + \tfrac{\pi}{2})$$

$$\frac{d}{dt}\cos\omega_0 t = -\omega_0\sin\omega_0 t = \omega_0\cos(\omega_0 t + \tfrac{\pi}{2})$$

Example #3: Impulse Response of an Ideal Lowpass Filter



$$h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}d\omega = \frac{\sin \omega_c t}{\pi t} = \frac{\omega_c}{\pi}\text{sinc}\left(\frac{\omega_c t}{\pi}\right)$$

Define: $\text{sinc}(\theta) = \frac{\sin \pi\theta}{\pi\theta}$

Questions:

1) Is this a causal system? No.

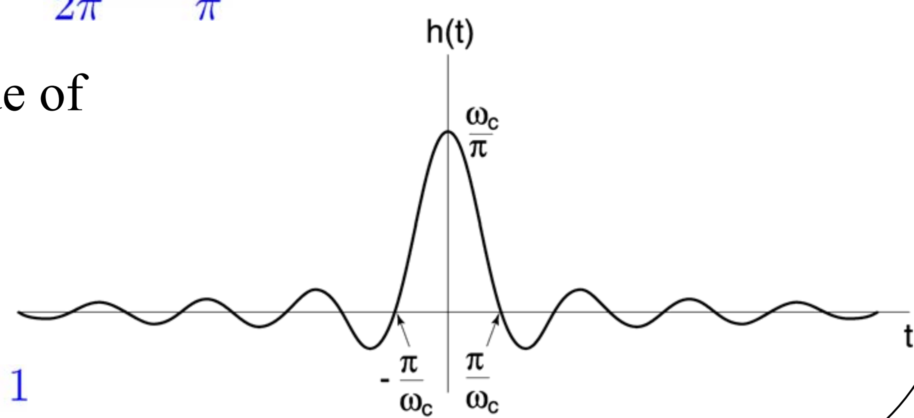
2) What is $h(0)$?

$$h(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(j\omega)d\omega = \frac{2\omega_c}{2\pi} = \frac{\omega_c}{\pi}$$

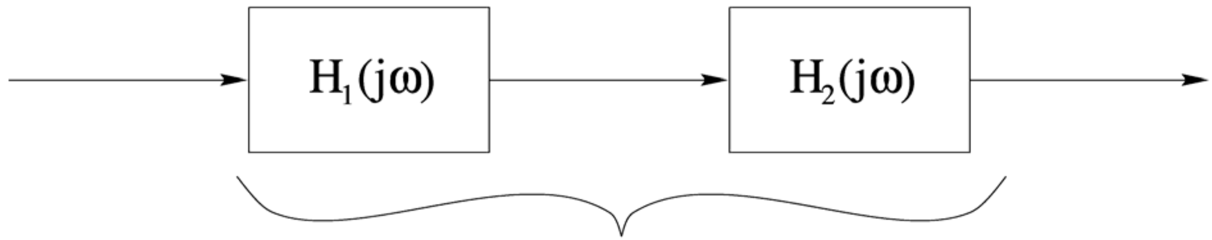
3) What is the steady-state value of the step response, i.e. $s(\infty)$?

$$s(t) = \int_{-\infty}^{t} h(t)dt$$

$$s(\infty) = \int_{-\infty}^{\infty} h(t)dt = H(j0) = 1$$

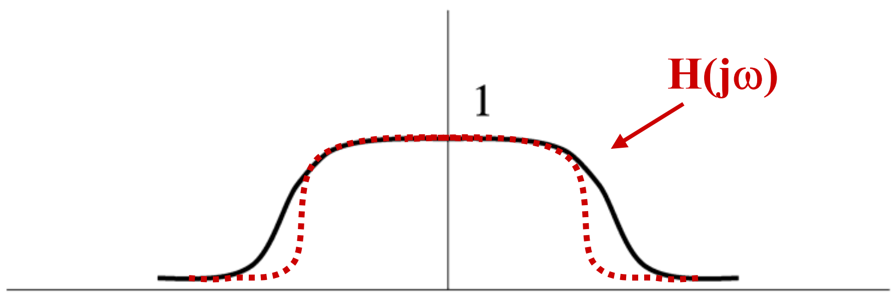


Example #4: Cascading filtering operations



$$H(j\omega) = H_1(j\omega)\, H_2(j\omega)$$

e.g. $H_1(j\omega) = H_2(j\omega)$



$\Longrightarrow$ $H(j\omega) = H_1^2(j\omega)$ has a sharper frequency selectivity

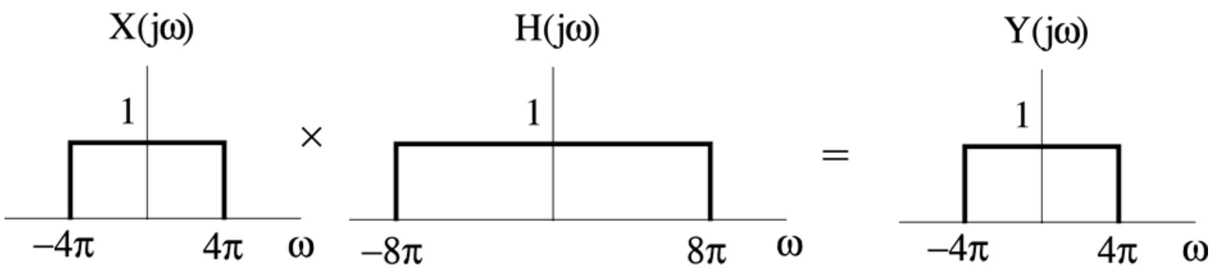
Example #5:

$$\underbrace{\frac{\sin 4\pi t}{\pi t}}_{x(t)} * \underbrace{\frac{\sin 8\pi t}{\pi t}}_{h(t)} = \quad ?$$

$$Y(j\omega) = X(j\omega)$$

$$\Rightarrow y(t) = x(t)$$

$$\Downarrow$$



Example #6:

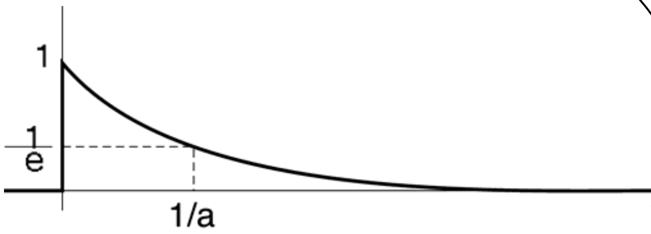
$$e^{-at^2} * e^{-bt^2} = \quad ? \quad \sqrt{\frac{\pi}{a+b}} \cdot e^{-(\frac{ab}{a+b})t^2}$$

$$\Downarrow$$

$$\sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} \times \sqrt{\frac{\pi}{b}} e^{-\frac{-\omega^2}{4b}} = \frac{\pi}{\sqrt{ab}} e^{-\frac{\omega^2}{4}\left(\frac{1}{a}+\frac{1}{b}\right)}$$

Gaussian × Gaussian = Gaussian ⇒ Gaussian * Gaussian = Gaussian

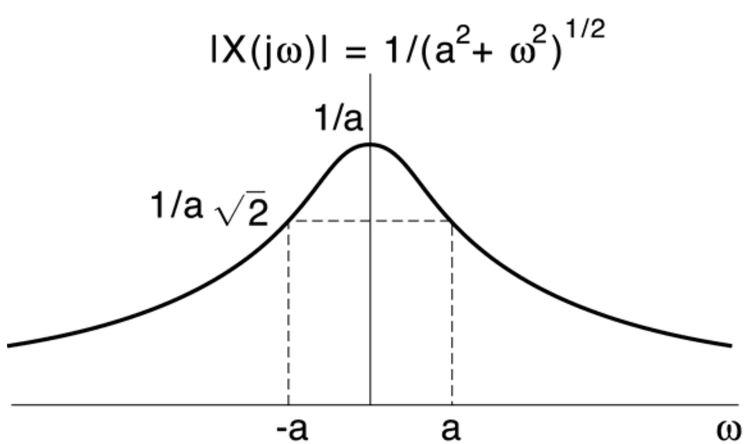
Example #2 from last lecture

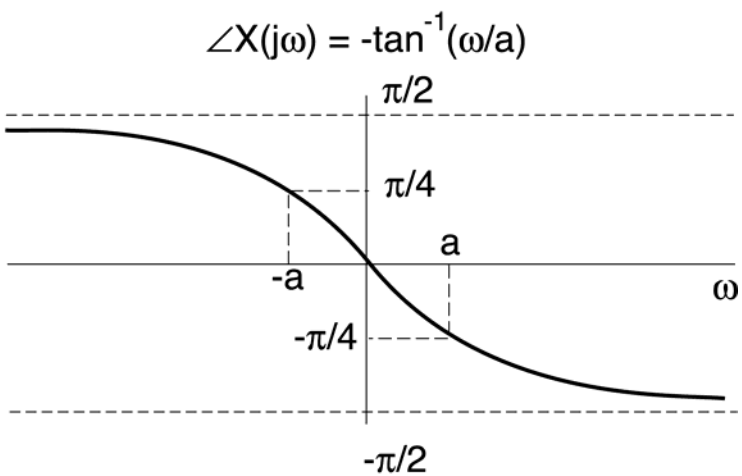


$$x(t) = e^{-at}u(t) \quad , \quad a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_0^{\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)}} dt$$

$$= -\left(\frac{1}{a+j\omega}\right) e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$





Example #7:

$$h(t) = e^{-t}u(t), \quad x(t) = e^{-2t}u(t)$$

$$y(t) = h(t) * x(t)$$

$$\Downarrow$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{1}{(1+j\omega)} \cdot \frac{1}{(2+j\omega)}$$

- a rational function of $j\omega$, ratio of polynomials of $j\omega$

$\Downarrow$ Partial fraction expansion

$$Y(j\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

$\Downarrow$ inverse FT

$$y(t) = [e^{-t} - e^{-2t}]u(t)$$

Example #8: LTI Systems Described by LCCDE's

(Linear-constant-coefficient differential equations)

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Using the Differentiation Property

$$\frac{d^k x(t)}{dt^k} \longleftrightarrow (j\omega)^k X(j\omega)$$

$\Downarrow$ Transform both sides of the equation

$$\sum_{k=0}^{N} a_k \cdot (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k \cdot (j\omega)^k X(j\omega)$$

$$\Downarrow$$

$$Y(j\omega) = \underbrace{\left[\frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}\right]}_{H(j\omega)} X(j\omega)$$

1) Rational, can use PFE to get $h(t)$
2) If $X(j\omega)$ is rational
   e.g. $x(t) = \sum c_l e^{-at} u(t)$
   then $Y(j\omega)$ is also rational

Parseval's Relation

$$\underbrace{\int_{-\infty}^{\infty} |x(t)|^2 dt}_{\substack{\text{Total energy} \\ \text{in the time-domain}}} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega}_{\substack{\text{Total energy} \\ \text{in the frequency-domain}}}$$

$\frac{1}{2\pi}|X(j\omega)|^2$ - Spectral density

Multiplication Property

FT is highly symmetric,

$$x(t) \overset{\mathcal{F}^{-1}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega, \; X(j\omega) \overset{\mathcal{F}}{=} \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

We already know that: $x(t) * y(t) \longleftrightarrow X(j\omega) \cdot Y(j\omega)$

Then it isn't a surprise that: $x(t) \cdot y(t) \longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\theta)Y(j(\omega - \theta))d\theta$$

Convolution in ω

— A consequence of *Duality*

Examples of the Multiplication Property

$$r(t) = s(t) \cdot p(t) \quad \longleftrightarrow \quad R(j\omega) = \frac{1}{2\pi}[S(j\omega) * P(j\omega)]$$

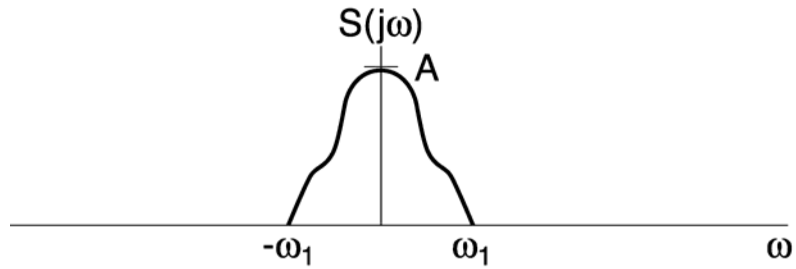
For $p(t) = \cos \omega_0 t \quad \longleftrightarrow \quad P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$

$$\Downarrow$$

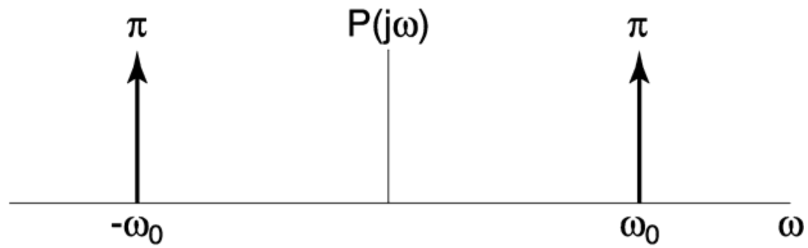
For any *s(t)* ...

$$R(j\omega) = \frac{1}{2}S(j(\omega - \omega_0)) + \frac{1}{2}S(j(\omega + \omega_0))$$

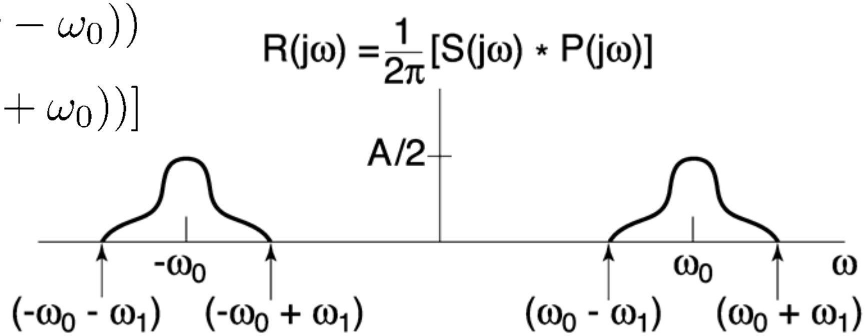
Example (continued)



$r(t) = s(t) \cdot \cos(\omega_0 t)$
Amplitude modulation (AM)



$$R(j\omega) \quad = \quad \frac{1}{2}[S(j(\omega - \omega_0)) + S(j(\omega + \omega_0))]$$



Drawn assuming:

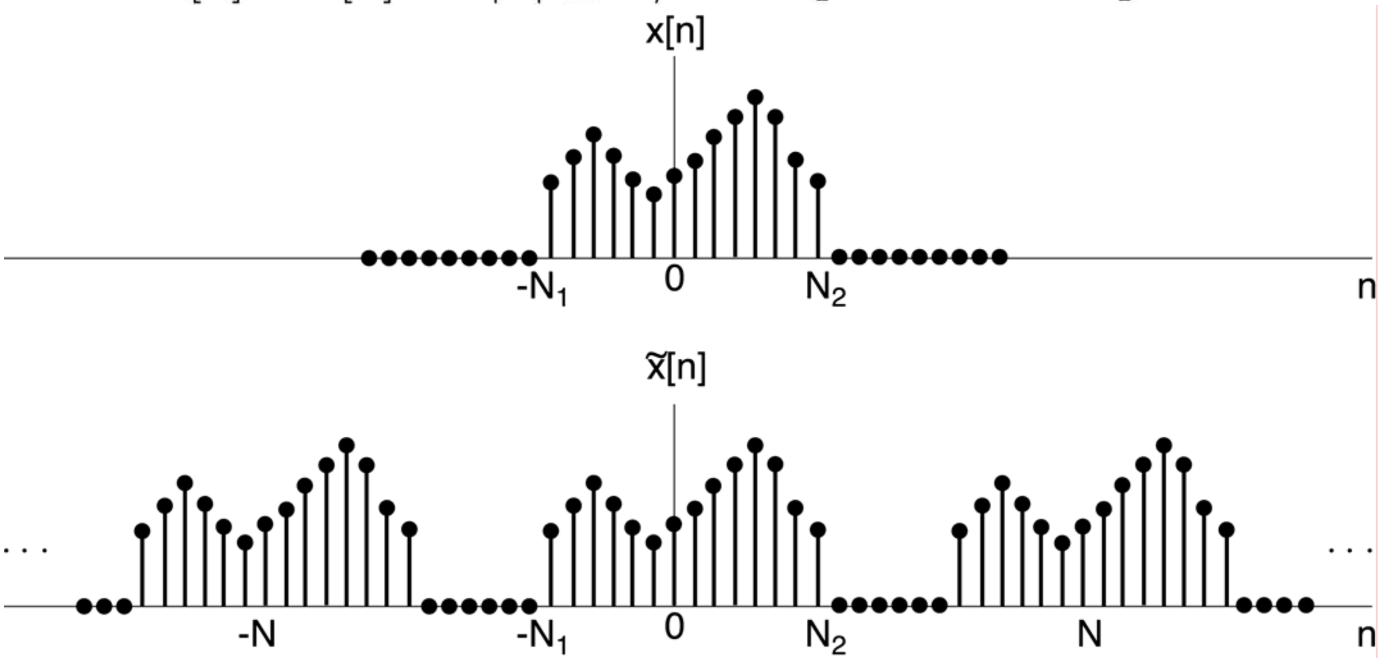
$$\omega_0 - \omega_1 > 0$$

$$i.e. \quad \omega_0 > \omega_1$$

The Discrete-Time Fourier Transform

Derivation: (Analogous to CTFT except $e^{j\omega n} = e^{j(\omega+2\pi)n}$)

- $x[n]$ - aperiodic and (for simplicity) of finite duration
- N is large enough so that $x[n] = 0$ if $|n| \geq N/2$
- $\tilde{x}[n] = x[n]$ for $|n| \leq N/2$ and periodic with period N



$$\tilde{x}[n] = x[n] \text{ for } any\ n \text{ as } N \rightarrow \infty$$

DTFT Derivation (Continued)

$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \ \omega_0 = \frac{2\pi}{N}$$ DTFS synthesis eq.

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\omega_0 n}$$ DTFS analysis eq.

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$$

Define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$ – periodic in ω with period 2π

$$\Downarrow$$

$$a_k = \frac{1}{N} X(e^{jk\omega_0})$$

DTFT Derivation (Home Stretch)

$$\tilde{x}[n] = \sum_{k=<N>} \underbrace{\frac{1}{N} X(e^{jk\omega_0})}_{a_k} e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0 \quad (*)$$

As $N \to \infty$: $\tilde{x}[n] \to x[n]$ for every n

$$\omega_0 \to 0, \ \sum \omega_0 \to \int d\omega$$

The sum in $(*)$ $\to$ an integral

$\Downarrow$ The DTFT Pair

$x[n] = \dfrac{1}{2\pi} \displaystyle\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$	Synthesis equation
$X(e^{j\omega}) = \displaystyle\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$	Analysis equation

Any 2π interval in ω