Signals and Systems

Fall 2003

Lecture #8

30 September 2003

1. Derivation of the CT Fourier Transform pair
2. Examples of Fourier Transforms
3. Fourier Transforms of Periodic Signals
4. Properties of the CT Fourier Transform

Fourier's Derivation of the CT Fourier Transform

- $x(t)$ - an aperiodic signal
  - view it as the limit of a periodic signal as $T \to \infty$

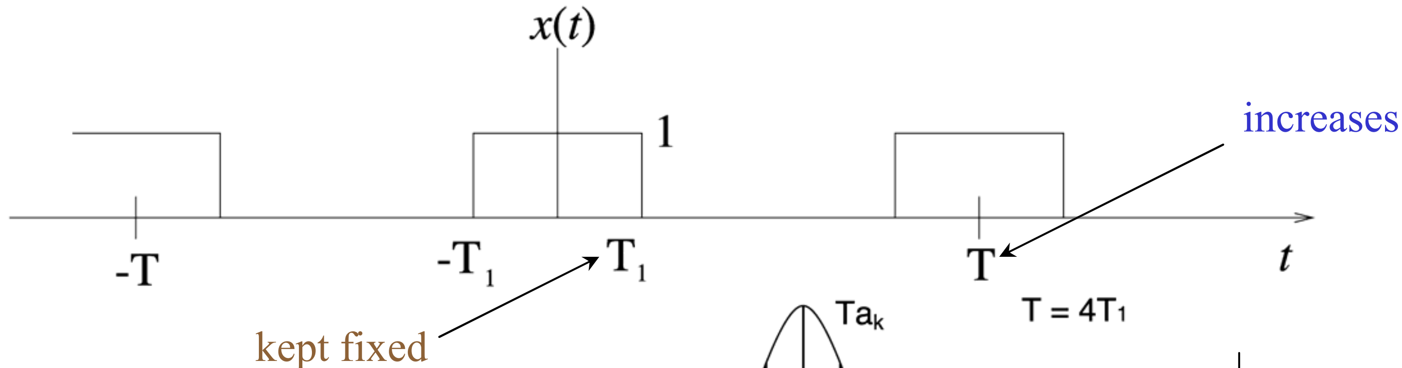
- For a periodic signal, the harmonic components are spaced $\omega_0 = 2\pi/T$ apart ...

- As $T \to \infty$, $\omega_0 \to 0$, and harmonic components are spaced closer and closer in frequency

$$\Downarrow$$

$$\text{Fourier series} \longrightarrow \text{Fourier integral}$$

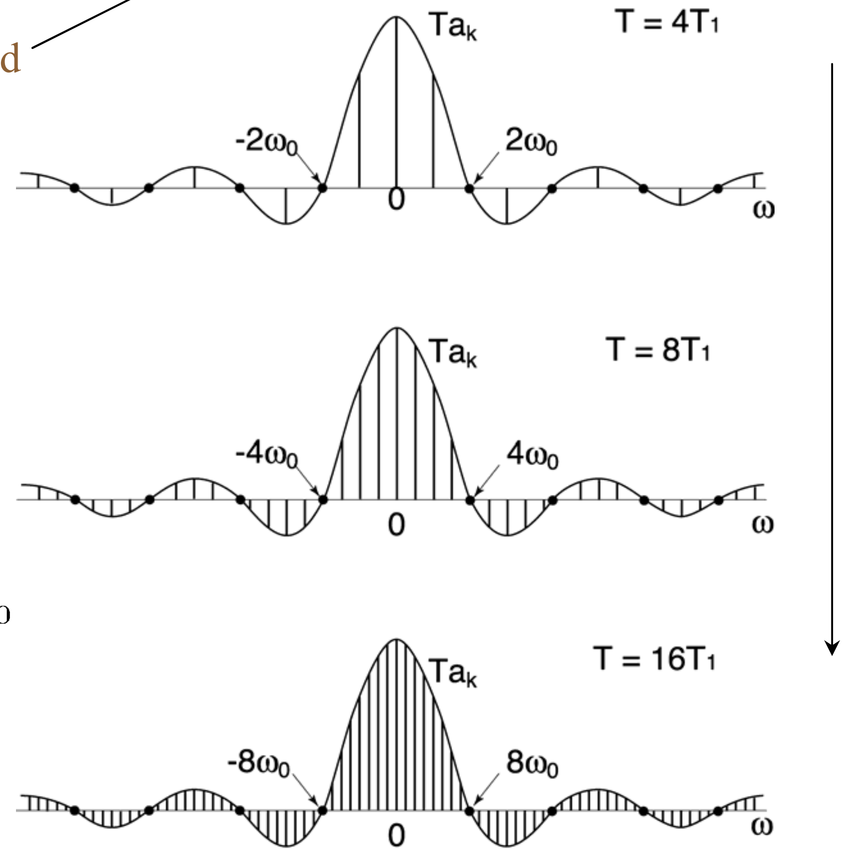
Motivating Example: Square wave



$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T}$$

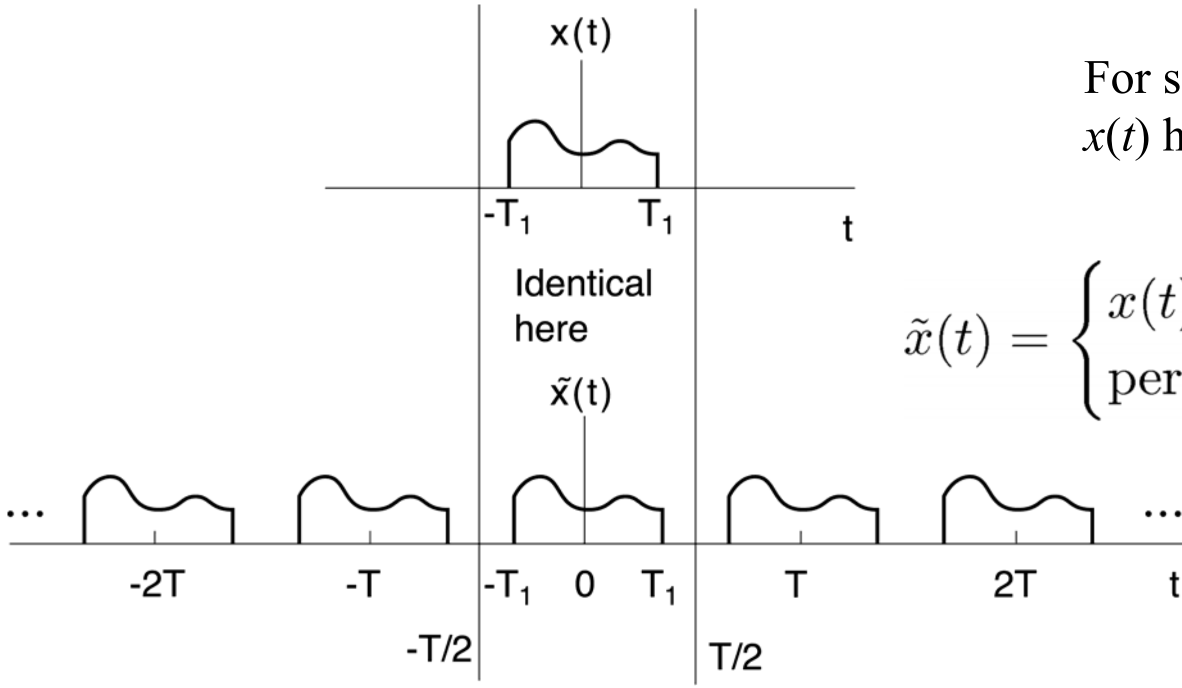
$$\Downarrow$$

$$Ta_k = \left.\frac{2\sin\omega T_1}{\omega}\right|_{\omega=k\omega_0}$$



Discrete frequency points become denser in ω as T increases

So, on with the derivation ...



For simplicity, assume *x*(*t*) has a finite duration.

$$\tilde{x}(t) = \begin{cases} x(t), & -\frac{T}{2} < t < \frac{T}{2} \\ \text{periodic}, & |t| > \frac{T}{2} \end{cases}$$

$$\text{As } T \to \infty,\ \tilde{x}(t) = x(t) \text{ for all } t$$

Derivation (continued)

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \left(\omega_0 = \frac{2\pi}{T}\right)$$

$$a_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \tilde{x}(t) e^{-jk\omega_0 t} dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt$$

$\uparrow$

$\tilde{x}(t) = x(t)$ in this interval

$$= \frac{1}{T}\int_{-\infty}^{\infty} x(t) e^{-jk\omega_0 t} dt \qquad (1)$$

If we define

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

then Eq.(1) $\Rightarrow$

$$a_k = \frac{X(jk\omega_0)}{T}$$

Derivation (continued)

Thus, for $-\frac{T}{2} < t < \frac{T}{2}$

$$x(t) = \tilde{x}(t) = \sum_{k=-\infty}^{\infty} \underbrace{\frac{1}{T} X(jk\omega_0)}_{a_k} e^{jk\omega_0 t}$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \omega_0 X(jk\omega_0) e^{jk\omega_0 t}$$

$$\Downarrow$$

As $T \to \infty$, $\sum \omega_0 \to \int d\omega$, we get the CT Fourier Transform pair

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{Synthesis equation}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{Analysis equation}$$

For what kinds of signals can we do this?

(1) It works also even if $x(t)$ is infinite duration, but satisfies:

a) Finite energy $$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$

In this case, there is *zero* energy in the error

$$e(t) = x(t) - \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \quad \text{Then} \quad \int_{-\infty}^{\infty} |e(t)|^2 dt = 0$$

b) Dirichlet conditions

(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = x(t)$ at points of continuity

(ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega =$ midpoint at discontinuity

(iii) Gibb's phenomenon

c) By allowing impulses in $x(t)$ or in $X(j\omega)$, we can represent even *more* signals

E.g. It allows us to consider *FT* for *periodic* signals

Example #1

(a) $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\Downarrow$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega$$

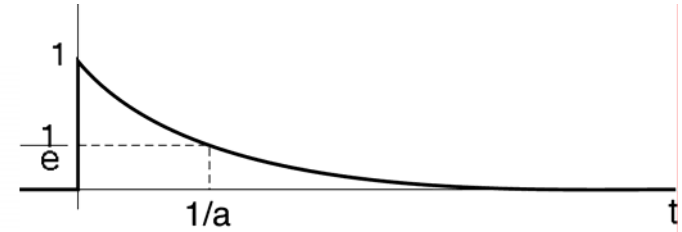
— Synthesis equation for $\delta(t)$

(b) $x(t) = \delta(t - t_0)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt$$

$$= e^{-j\omega t_0}$$

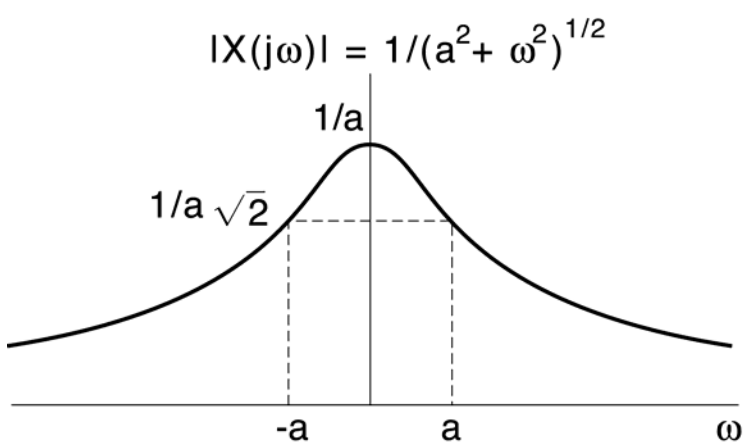
Example #2: Exponential function



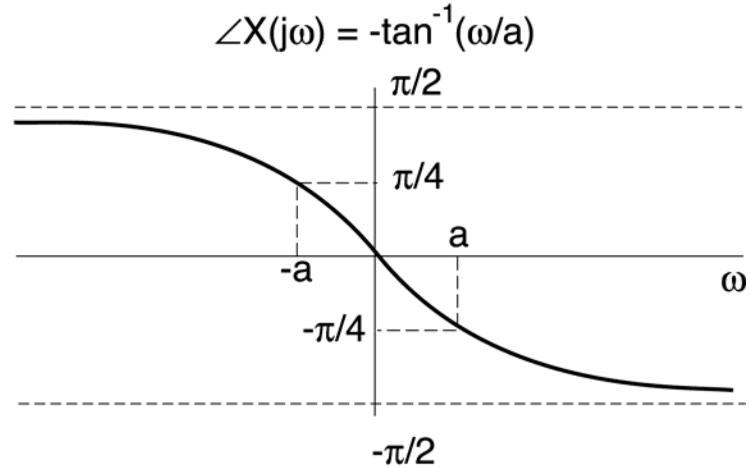
$$x(t) = e^{-at}u(t), a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}} dt$$

$$= -\left(\frac{1}{a+j\omega}\right) e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \frac{1}{a+j\omega}$$



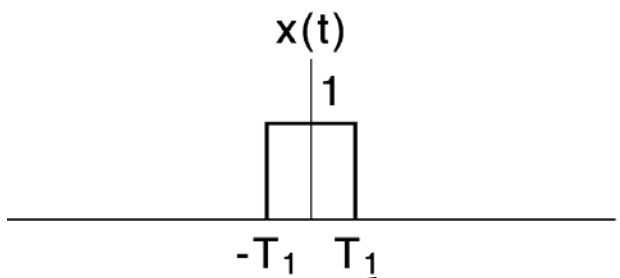
Even symmetry

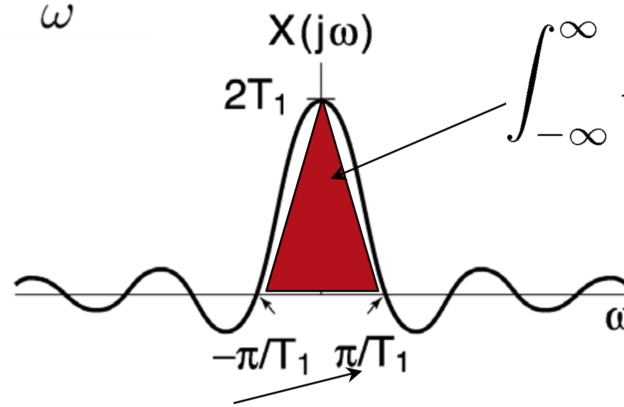


Odd symmetry

Example #3: A square pulse in the time-domain

$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = \frac{2\sin\omega T_1}{\omega}$$





$$\int_{-\infty}^{\infty} X(j\omega) d\omega$$

Note the inverse relation between the two widths ⇒ Uncertainty principle

Useful facts about CTFT's

$$X(0) = \int_{-\infty}^{\infty} x(t) dt$$

Example above: $\int_{-\infty}^{\infty} x(t)dt = 2T_1 = X(0)$

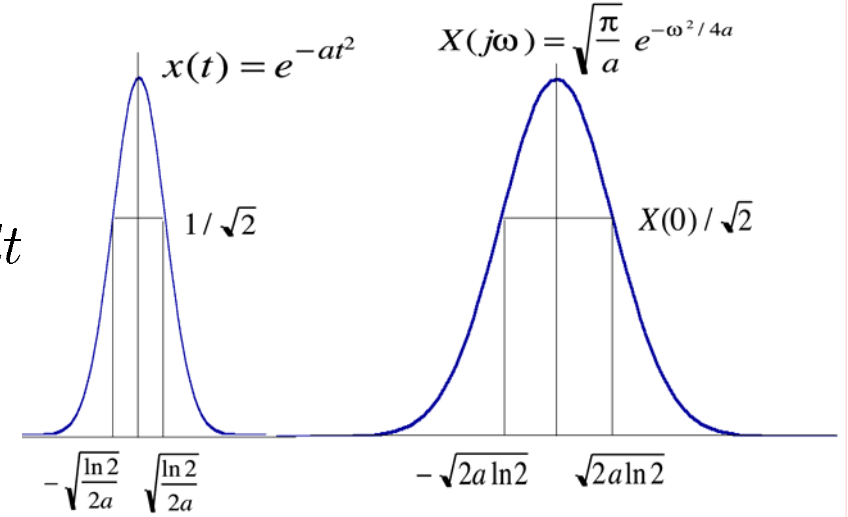
$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) d\omega$$

$$\text{Ex. above: } x(0) = 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) d\omega$$

$$= \frac{1}{2\pi} \times \text{(Area of the triangle)}$$

Example #4: $x(t) = e^{-at^2}$ — A Gaussian, important in probability, optics, etc.

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt \\
&= \int_{-\infty}^{\infty} e^{-a\left[t^2 + j\frac{\omega}{a}t + \left(\frac{j\omega}{2a}\right)^2\right] + a\left(\frac{j\omega}{2a}\right)^2} dt \\
&= \underbrace{\left[\int_{-\infty}^{\infty} e^{-a\left(t+\frac{j\omega}{2a}\right)^2} dt\right]}_{\sqrt{\pi}/a} \cdot e^{-\frac{\omega^2}{4a}} \\
&= \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}
\end{aligned}
$$



Also a Gaussian!

(Pulse width in t)•(Pulse width in ω)

$\Rightarrow \Delta t \cdot \Delta\omega \sim (1/a^{1/2}) \cdot (a^{1/2}) = 1$

Uncertainty Principle! Cannot make *both* Δt and $\Delta\omega$ arbitrarily small.

CT Fourier Transforms of Periodic Signals

Suppose

$$X(j\omega) = \delta(\omega - \omega_0)$$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

— periodic in *t* with frequency ω_o

That is

$$\boxed{e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)}$$

— All the energy is concentrated in one frequency — ω_o

More generally

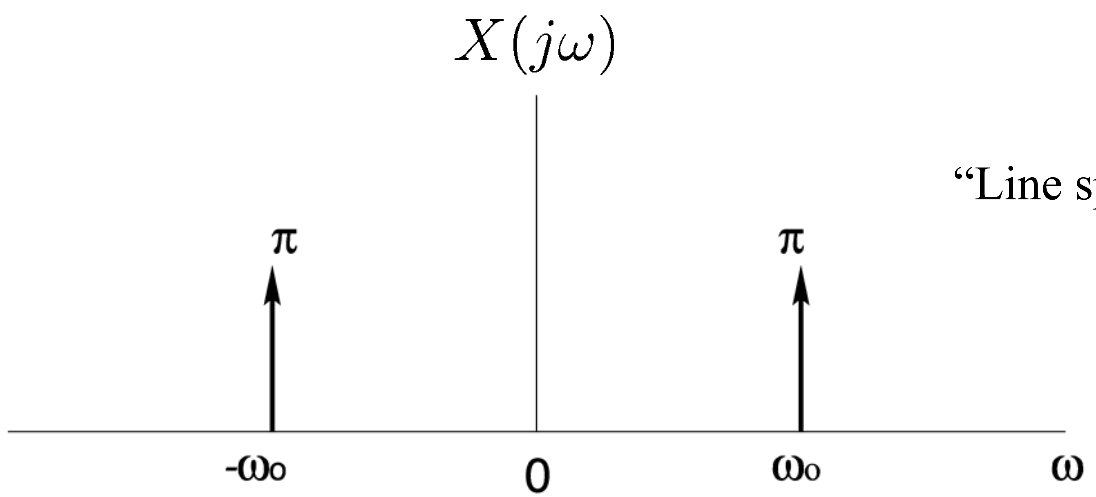
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Example #4:

$$x(t) = \cos \omega_0 t = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

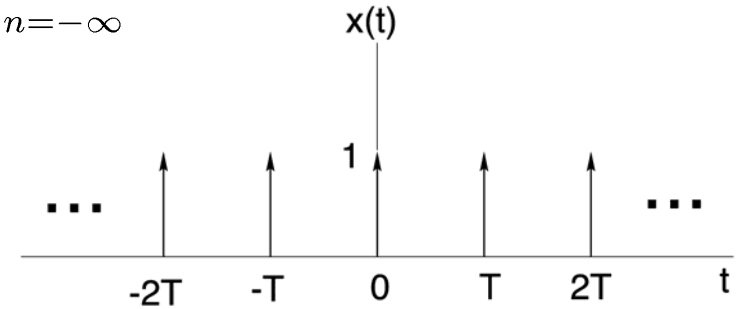
$$\updownarrow$$

$$X(j\omega) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$$



"Line spectrum"

Example #5: $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$ — Sampling function

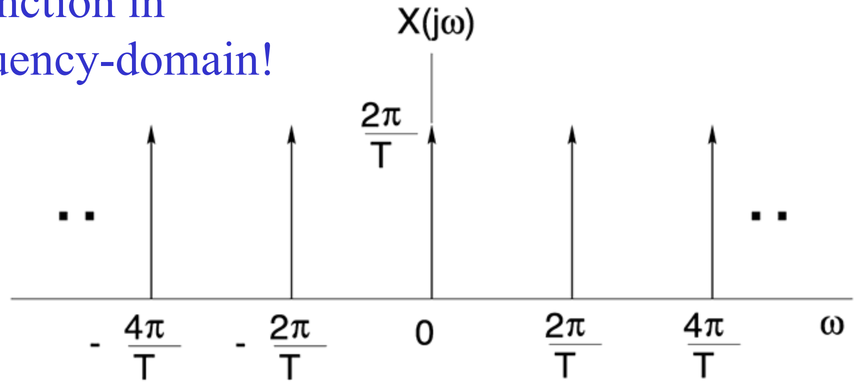


$$x(t) \leftrightarrow a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}$$

$$\Downarrow$$

$$X(j\omega) = \sum_{n=-\infty}^{\infty} \underbrace{\frac{2\pi}{T}}_{2\pi a_k} \delta(\omega - \underbrace{\frac{k2\pi}{T}}_{k\omega_0})$$

Same function in
the frequency-domain!



Note: (period in t) T
$\Leftrightarrow$ (period in ω) $2\pi/T$
Inverse relationship again!

Properties of the CT Fourier Transform

1) Linearity $$ax(t) + by(t) \leftrightarrow aX(j\omega) + bY(j\omega)$$

2) Time Shifting $$x(t - t_0) \leftrightarrow e^{-j\omega t_0} X(j\omega)$$

Proof: $$\int_{-\infty}^{\infty} x(\underbrace{t - t_0}_{t'}) e^{-j\omega t} dt = e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} x(t') e^{-j\omega t'} dt'}_{X(j\omega)}$$

FT magnitude unchanged

$$|e^{-j\omega t_0} X(j\omega)| = |X(j\omega)|$$

Linear change in *FT* phase

$$\angle(e^{-j\omega t_0} X(j\omega)) = \angle X(j\omega) - \omega t_0$$

Properties (continued)

3) Conjugate Symmetry

$$x(t) \text{ real} \leftrightarrow X(-j\omega) = X^*(j\omega)$$

$$\Downarrow$$

$$|X(-j\omega)| = |X(j\omega)|$$ *Even*

$$\angle X(-j\omega) = -\angle X(j\omega)$$ *Odd*

$$Re\{X(-j\omega)\} = Re\{X(j\omega)\}$$ *Even*

$$Im\{X(-j\omega)\} = -Im\{X(j\omega)\}$$ *Odd*

The Properties Keep on Coming ...

4) Time-Scaling

$$x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

E.g. $a > 1 \rightarrow at > t$
compressed in time $\Leftrightarrow$
stretched in frequency

$$\Downarrow \; a = -1$$

$$x(-t) \longleftrightarrow X(-j\omega)$$

$$\Downarrow$$

a) $x(t)$ real and even

$$x(t) = x(-t)$$

$$\Rightarrow X(j\omega) = X(-j\omega) = X^*(j\omega) - \text{ Real \& even}$$

b) $x(t)$ real and odd

$$x(t) = -x(-t)$$

$$\Rightarrow X(j\omega) = -X(-j\omega) = -X^*(j\omega) - \text{ Purely imaginary \& odd}$$

c)

$$\begin{array}{ccccc} X(j\omega) & = & Re\{X(j\omega)\} & + & jIm\{X(j\omega)\} \\ & & \uparrow & & \uparrow \\ \textbf{For real } x(t) & = & Ev\{x(t)\} & + & Od\{x(t)\} \end{array}$$