

# Signals and Systems

Fall 2003

Lecture #23

4 December 2003

1. Geometric Evaluation of  $z$ -Transforms and DT Frequency Responses
2. First- and Second-Order Systems
3. System Function Algebra and Block Diagrams
4. Unilateral  $z$ -Transforms

## Geometric Evaluation of a Rational z-Transform

### Example #1:

$$X_1(z) = z - a - \text{A first-order zero}$$

### Example #2:

$$X_2(z) = \frac{1}{z - a} - \text{A first-order pole}$$

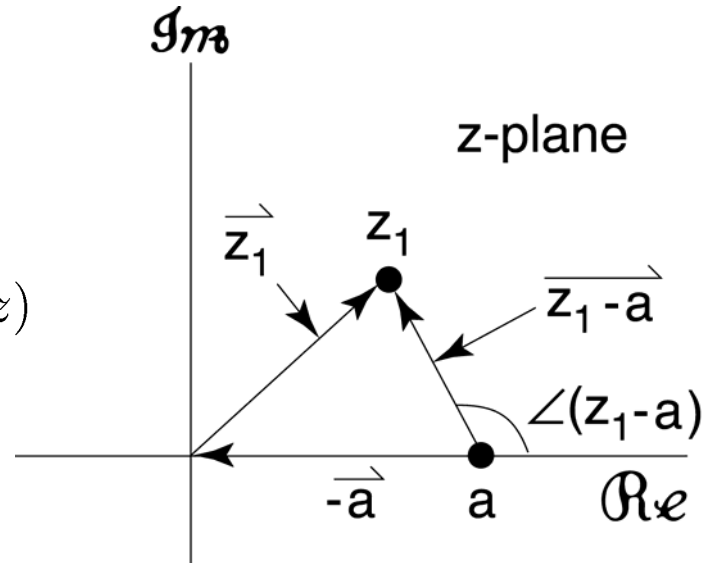
$$|X_2(z)| = \frac{1}{|X_1(z)|}, \quad \angle X_2(z) = -\angle X_1(z)$$

### Example #3:

$$X(z) = M \frac{\prod_{i=1}^R (z - \beta_i)}{\prod_{j=1}^P (z - \alpha_j)}$$

$$|X(z)| = |M| \frac{\prod_{i=1}^R |z - \beta_i|}{\prod_{j=1}^P |z - \alpha_j|}$$

$$\angle X(z) = \angle M + \sum_{i=1}^R \angle(z - \beta_i) - \sum_{j=1}^P \angle(z - \alpha_j)$$



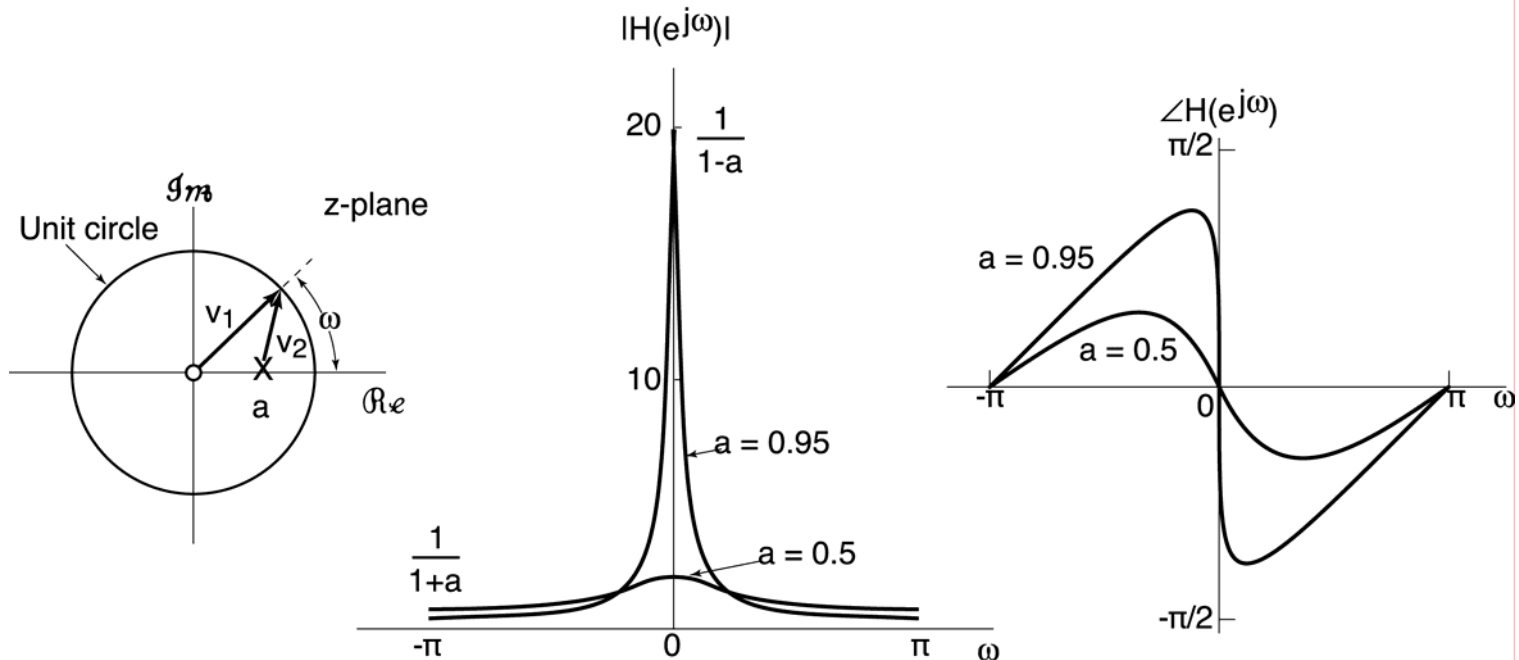
All same as  
in s-plane

# Geometric Evaluation of DT Frequency Responses

First-Order System — one *real* pole

$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|$$

$$h[n] = a^n u[n], \quad |a| < 1$$



$$H(e^{j\omega}) = \frac{v_1}{v_2}, \quad |H(e^{j\omega})| = \frac{|v_1|}{|v_2|} = \frac{1}{|v_2|}, \quad \angle H(e^{j\omega}) = \angle v_1 - \angle v_2 = \omega - \angle v_2$$

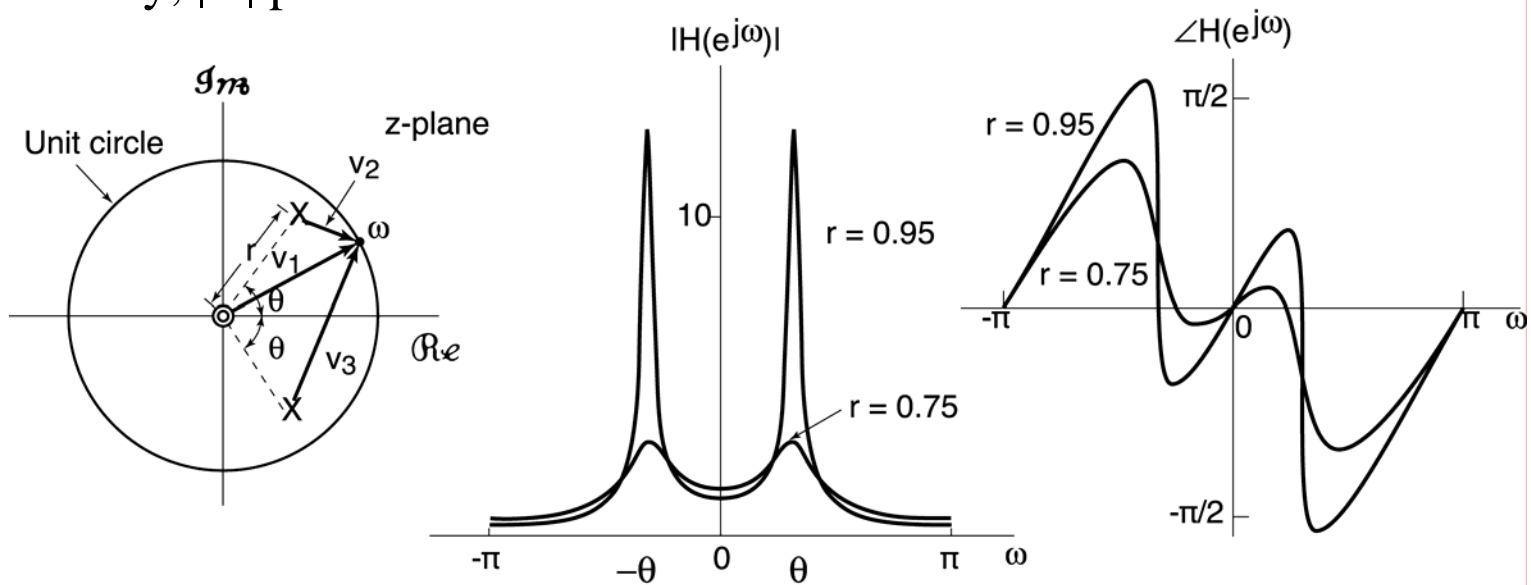
## Second-Order System

Two poles that are a complex conjugate pair ( $z_1 = re^{j\theta} = z_2^*$ )

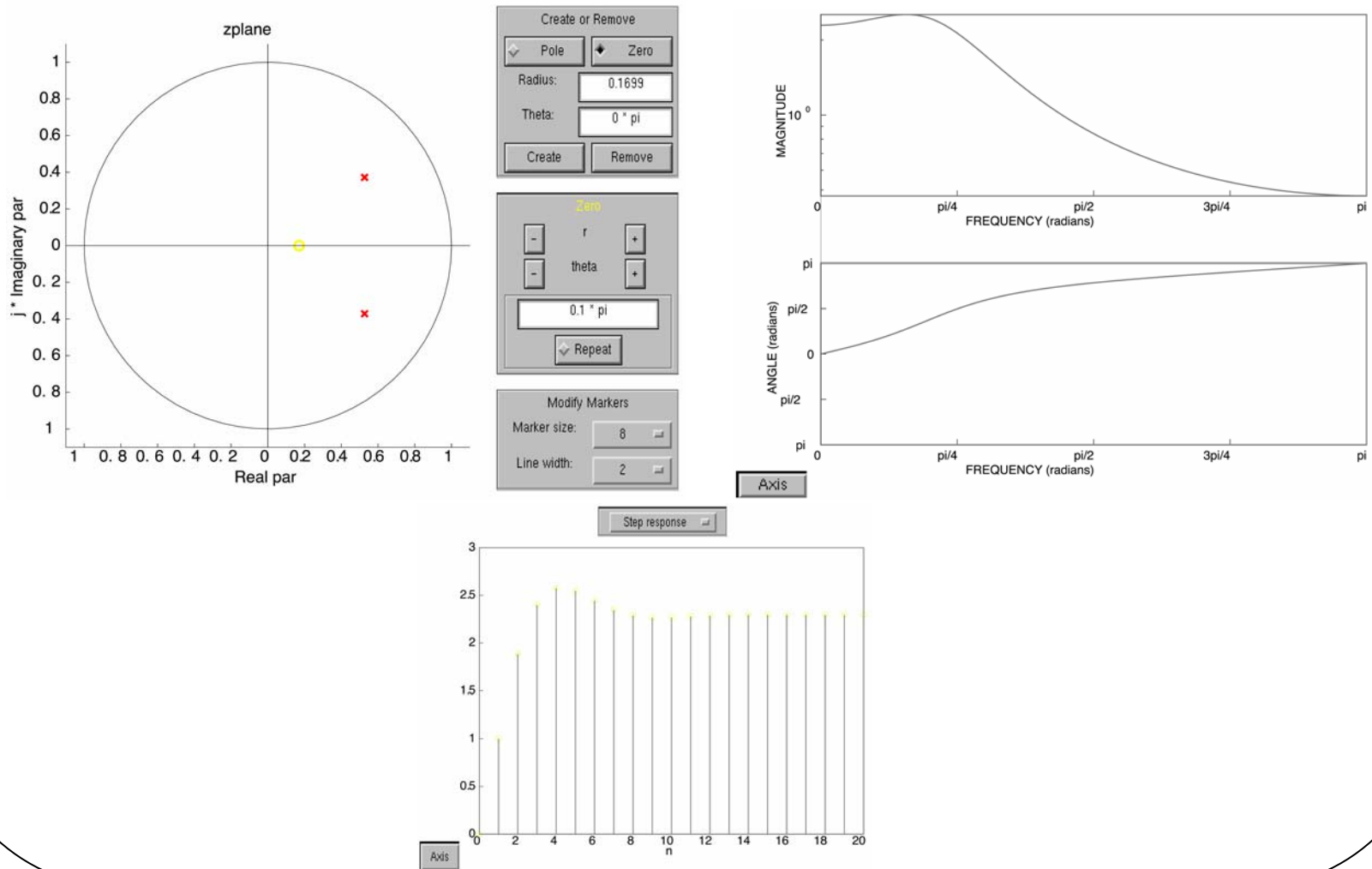
$$H(z) = \frac{z^2}{(z - z_1)(z - z_2)} = \frac{1}{1 - (2r \cos \theta)z^{-1} + r^2 z^{-2}}, \quad 0 < r < 1, \quad 0 \leq \theta \leq \pi$$

$$|H(e^{j\omega})| = \frac{1}{|(e^{j\omega} - re^{j\theta})(e^{j\omega} - re^{-j\theta})|}, \quad h[n] = r^n \frac{\sin[(n+1)\theta]}{\sin \theta} u[n]$$

Clearly,  $|H|$  peaks near  $\omega = \pm\theta$



# Demo: DT pole-zero diagrams, frequency response, vector diagrams, and impulse- & step-responses



## DT LTI Systems Described by LCCDEs

$$\sum_{k=0}^N a_k y[n - k] = \sum_{k=0}^M b_k x[n - k]$$

Use the time-shift property

$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{k=0}^M b_k z^{-k} X(z)$$

⇓

$$Y(z) = H(z)X(z)$$

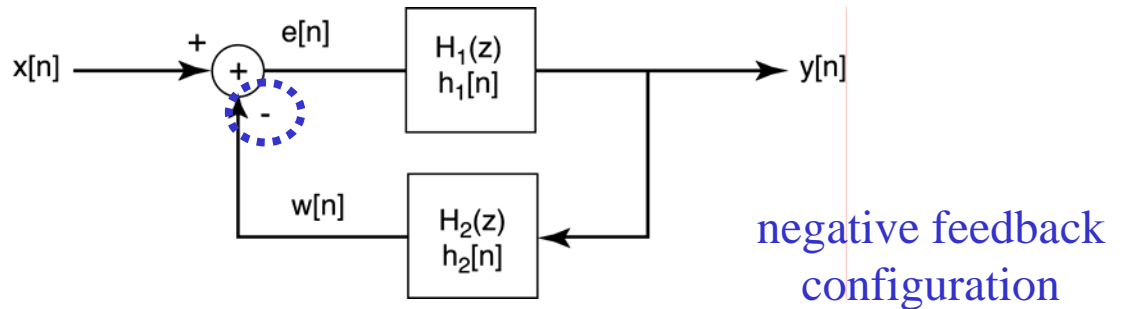
$$H(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} \quad \text{— Rational}$$

ROC: Depends on Boundary Conditions, left-, right-, or two-sided.

For Causal Systems  $\Rightarrow$  ROC is outside the outermost pole

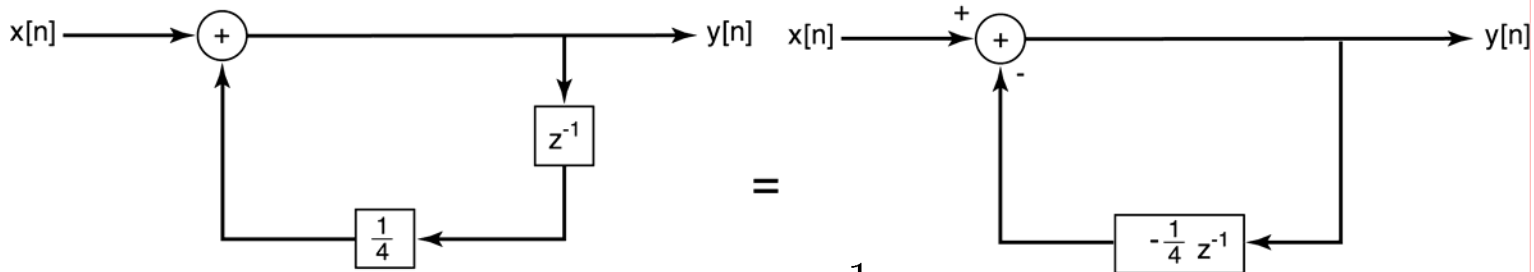
# System Function Algebra and Block Diagrams

Feedback System  
(causal systems)



$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)}$$

**Example #1:**



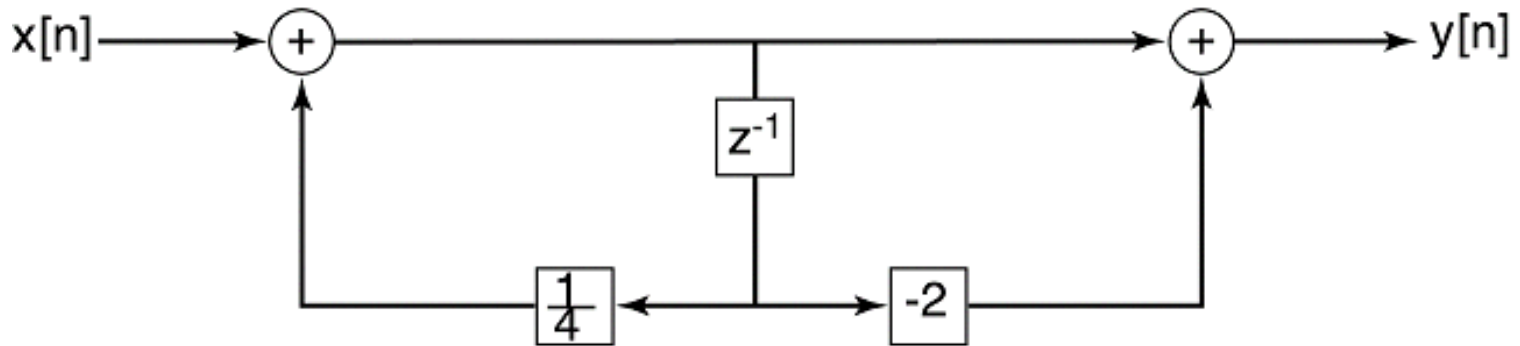
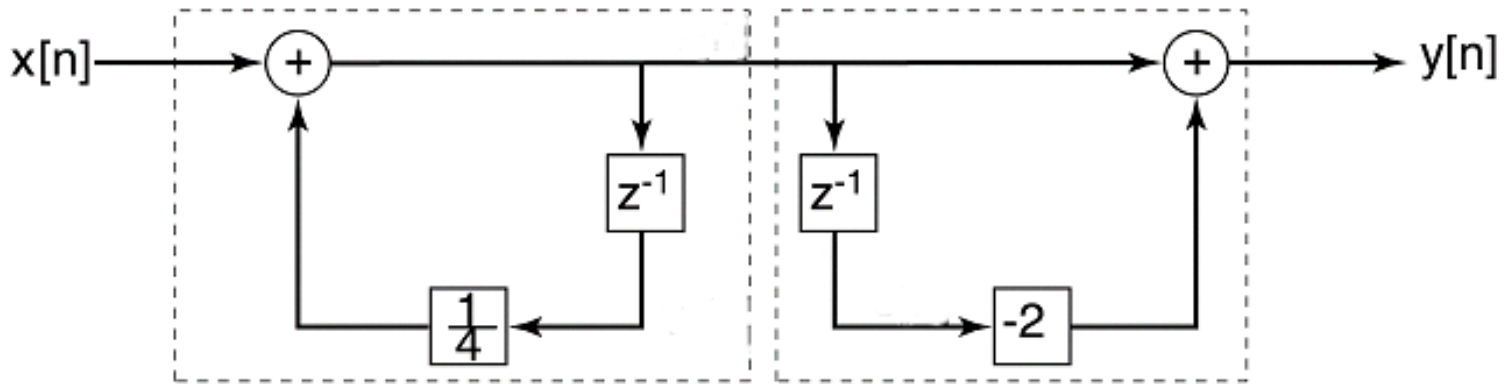
$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}}$$

$z^{-1} \Leftrightarrow \text{D}$   
Delay

$$y[n] = \frac{1}{4}y[n-1] + x[n]$$

**Example #2:**

$$H(z) = \frac{1 - 2z^{-1}}{1 - \frac{1}{4}z^{-1}} = \left( \frac{1}{1 - \frac{1}{4}z^{-1}} \right) (1 - 2z^{-1}) \text{ --- Cascade of two systems}$$



## Unilateral z-Transform

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Note:

- (1) If  $x[n] = 0$  for  $n < 0$ , then  $\mathcal{X}(z) = X(z)$
- (2) UZT of  $x[n] = \text{BZT of } x[n]u[n] \Rightarrow \text{ROC } \textit{always} \text{ outside a circle}$   
and *includes*  $z = \infty$
- (3) For causal LTI systems,  $\mathcal{H}(z) = H(z)$

## Properties of Unilateral z-Transform

Many properties are analogous to properties of the BZT e.g.

- Convolution property (for  $x_1[n < 0] = x_2[n < 0] = 0$ )

$$x_1[n] * x_2[n] \xleftrightarrow{\mathcal{U}Z} \mathcal{X}_1(z)\mathcal{X}_2(z)$$

- But there are important differences. For example, *time-shift*

$$y[n] = x[n - 1] \longleftrightarrow \mathcal{Y}(z) = x[-1] + z^{-1}\mathcal{X}(z)$$

Derivation:

$$\begin{aligned} \mathcal{Y}(z) &= \sum_{n=0}^{\infty} y[n]z^{-n} = \sum_{n=0}^{\infty} x[n - 1]z^{-n} = x[-1] + \sum_{n=1}^{\infty} x[n - 1]z^{-n} \\ &= x[-1] + z^{-1} \underbrace{\sum_{m=0}^{\infty} x[m]z^{-m}}_{\mathcal{X}(z)} \end{aligned}$$

Initial condition

## Use of UZTs in Solving Difference Equations with Initial Conditions

$$y[n] + 2y[n - 1] = x[n]$$

$$y[-1] = \beta, \quad x[n] = \alpha u[n] \longleftrightarrow \frac{\alpha}{1 - z^{-1}}$$

UZT of Difference Equation

$$\mathcal{Y}(z) + 2 \overbrace{[\beta + z^{-1}\mathcal{Y}(z)]}^{\mathcal{UZ}\{y[n-1]\}} = \frac{\alpha}{1 - z^{-1}}$$

$$\mathcal{Y}(z) = - \underbrace{\frac{2\beta}{1 + 2z^{-1}}}_{ZIR} + \underbrace{\frac{\alpha}{(1 + 2z^{-1})(1 - z^{-1})}}_{ZSR}$$

**ZIR** — Output purely due to the initial conditions,

**ZSR** — Output purely due to the input.

## Example (continued)

$\beta = 0 \Rightarrow$  System is initially at rest:

$$\text{ZSR} \quad \mathcal{Y}(z) = \mathcal{H}(z)\mathcal{X}(z) = \underbrace{\frac{1}{1+2z^{-1}}}_{\mathcal{H}(z)} \underbrace{\frac{\alpha}{1-z^{-1}}}_{\mathcal{X}(z)}$$

$$\mathcal{H}(z) = H(z) = \frac{1}{1+2z^{-1}}$$

$\alpha = 0 \Rightarrow$  Get response to initial conditions

$$\text{ZIR} \quad \mathcal{Y}(z) = -\frac{2\beta}{1+2z^{-1}}$$

$$y[n] = -2\beta(-2)^n u[n]$$