Signals and Systems

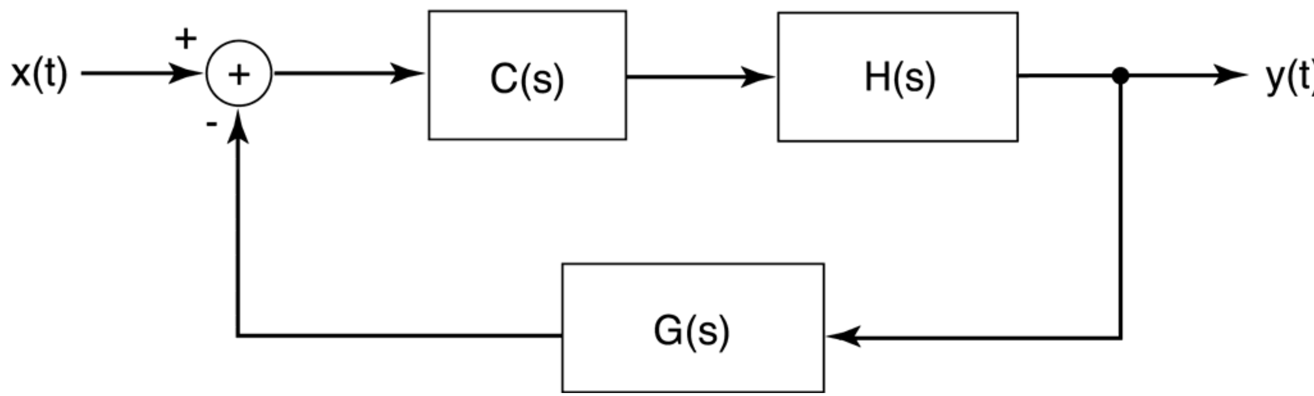
Fall 2003

Lecture #21

25 November 2003

1. Feedback
    a) Root Locus
    b) Tracking
    c) Disturbance Rejection
    d) The Inverted Pendulum
2. Introduction to the Z-Transform

The Concept of a Root Locus

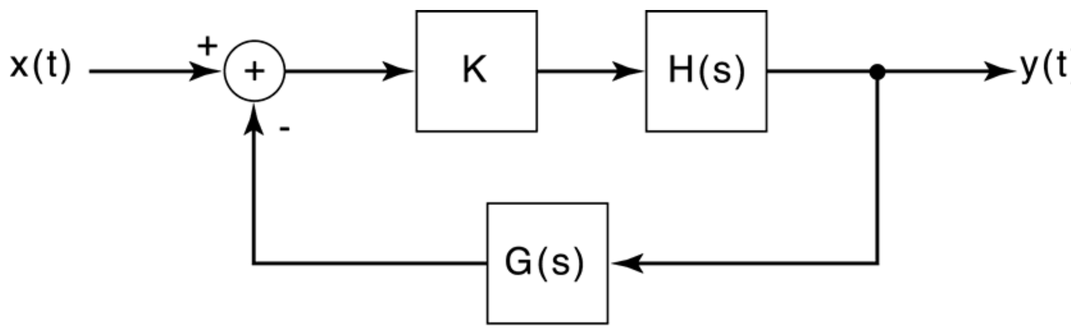


$$Q(s) = \frac{C(s)H(s)}{1 + C(s)G(s)H(s)}$$

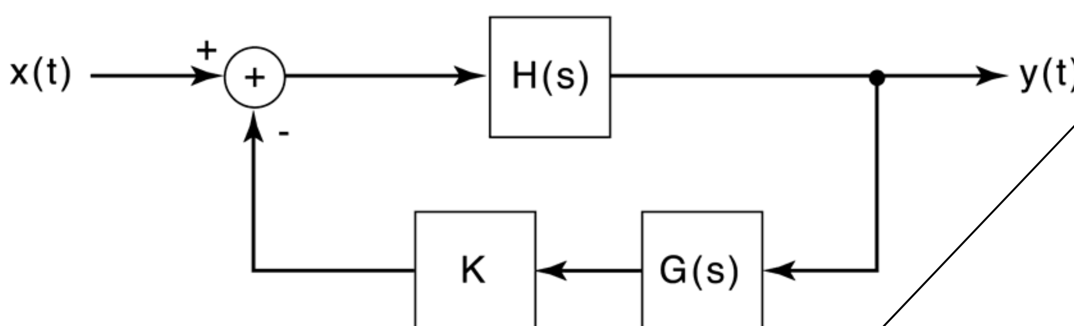
- *C*(s), *G*(*s*) — Designed with one or more free parameters
- Question: How do the closed-loop poles move as we vary these parameters? — Root locus of 1+ *C*(s)*G*(*s*)*H*(*s*)

The "Classical" Root Locus Problem

$C(s) = K$ — a simple linear amplifier



$$Q(s) = \frac{KH(s)}{1 + KH(s)G(s)}$$



$$Q(s) = \frac{H(s)}{1 + KH(s)G(s)}$$

Closed-loop poles are the same.

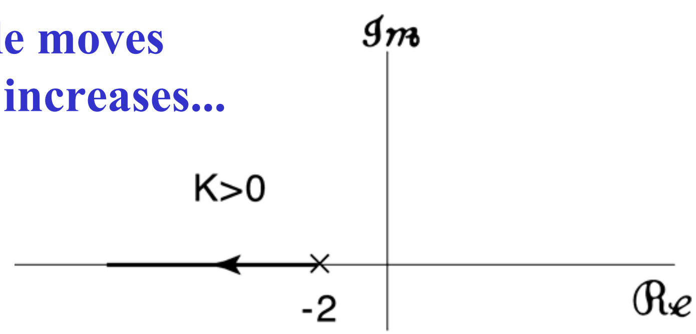
A Simple Example

$$H(s) = \frac{1}{s+2}, \quad G(s) = 1$$

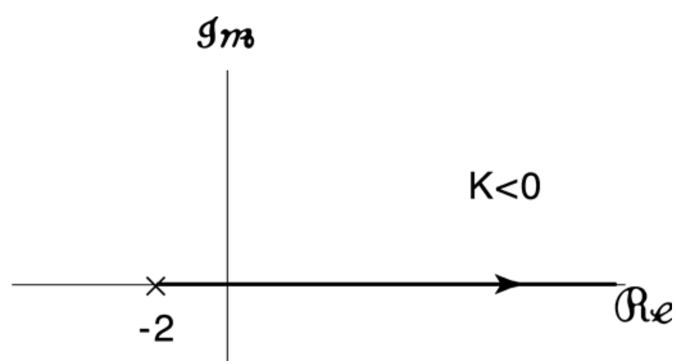
(a) $Q(s) = \frac{\frac{K}{s+2}}{1+\frac{K}{s+2}} = \frac{K}{s+K+2}$ (b) $Q(s) = \frac{\frac{1}{s+2}}{1+\frac{K}{s+2}} = \frac{1}{s+K+2}$

In either case, pole is at $s_o = -2 - K$

Sketch where pole moves as |K| increases...



Becomes *more* stable



Becomes *less* stable

What Happens More Generally ?

- For simplicity, suppose there is no pole-zero cancellation in $G(s)H(s)$

$$Q(s) = \frac{H(s)}{1 + KG(s)H(s)}$$

Closed-loop poles are the solutions of

$$1 + KG(s)H(s) = 0$$

That is $$G(s)H(s) = -\frac{1}{K}$$

— Difficult to solve explicitly for solutions given any *specific* value of K, unless $G(s)H(s)$ is second-order or lower.

— Much easier to plot the *root locus*, the values of s that are solutions for *some* value of K, because:

1) It is easier to find the roots in the limiting cases for $K = 0, \pm\infty$.
2) There are rules on how to connect between these limiting points.

Rules for Plotting Root Locus

$$G(s)H(s) = -\frac{1}{K}$$

- End points

— At $K = 0$, $G(s_o)H(s_o) = \infty$

$\Rightarrow$ s_o are *poles* of the open-loop system function $G(s)H(s)$.

— At $|K| = \infty$, $G(s_o)H(s_o) = 0$

$\Rightarrow$ s_o are *zeros* of the open-loop system function $G(s)H(s)$. Thus:

Rule #1:

A root locus starts (at $K = 0$) from a *pole* of $G(s)H(s)$ and ends (at $|K| = \infty$) at a *zero* of $G(s)H(s)$.

Question: What if the number of *poles* $\neq$ the number of *zeros*?

Answer: Start or end at $\pm\infty$.

Rule #2: Angle criterion of the root locus

$$G(s)H(s) = \underbrace{-\frac{1}{K}}_{\text{real number}}$$

- Thus, s_0 is a pole for some *positive* value of K if:

$$K \geq 0 \Rightarrow \angle G(s_0)H(s_0) = (2n+1)\pi;$$

In this case, s_0 is a pole if $K = 1/|G(s_0)\, H(s_0)|$.

- Similarly s_0 is a pole for some *negative* value of K if:

$$K \leq 0 \Rightarrow \angle G(s_0)H(s_0) = 2n\pi$$

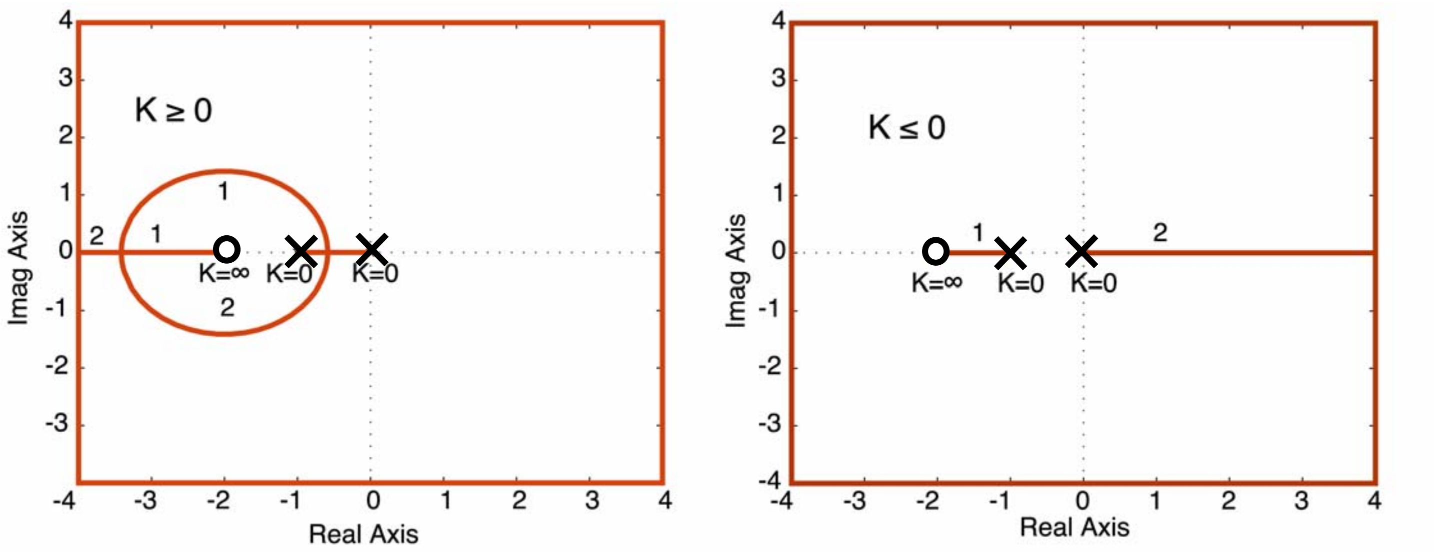
In this case, s_0 is a pole if $K = -1/|G(s_0)\, H(s_0)|$.

Example of Root Locus.

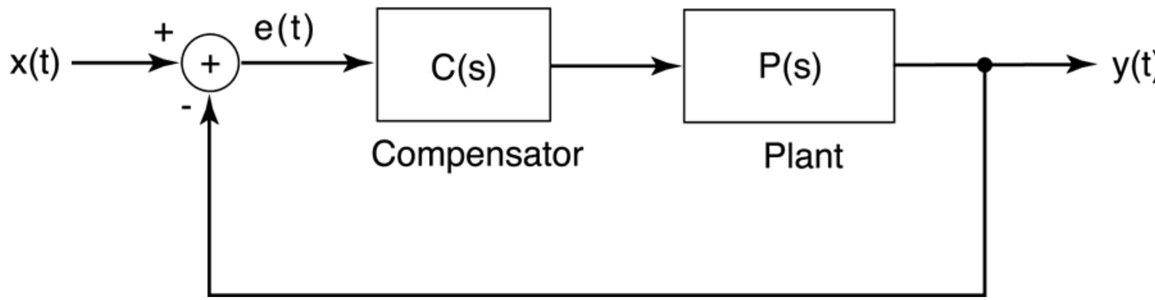
$$G(s)H(s) = \frac{s+2}{s(s+1)}$$

One zero at -2,
two poles at 0, -1.

$$rlocus(G(s)H(s))$$



Tracking



In addition to stability, we may want good tracking behavior, i.e.

$$e(t) = x(t) - y(t) \approx 0 \text{ as } t \rightarrow \infty$$

for at least some set of input signals.

$$E(s) = \frac{1}{1 + C(s)P(s)} X(s)$$

$$\Downarrow$$

$$E(j\omega) = \frac{1}{1 + C(j\omega)P(j\omega)} X(j\omega)$$

We want $C(j\omega)P(j\omega)$ to be *large* in frequency bands in which we want good tracking

Tracking (continued)

Using the final-value theorem

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1 + C(s)P(s)} X(s)$$

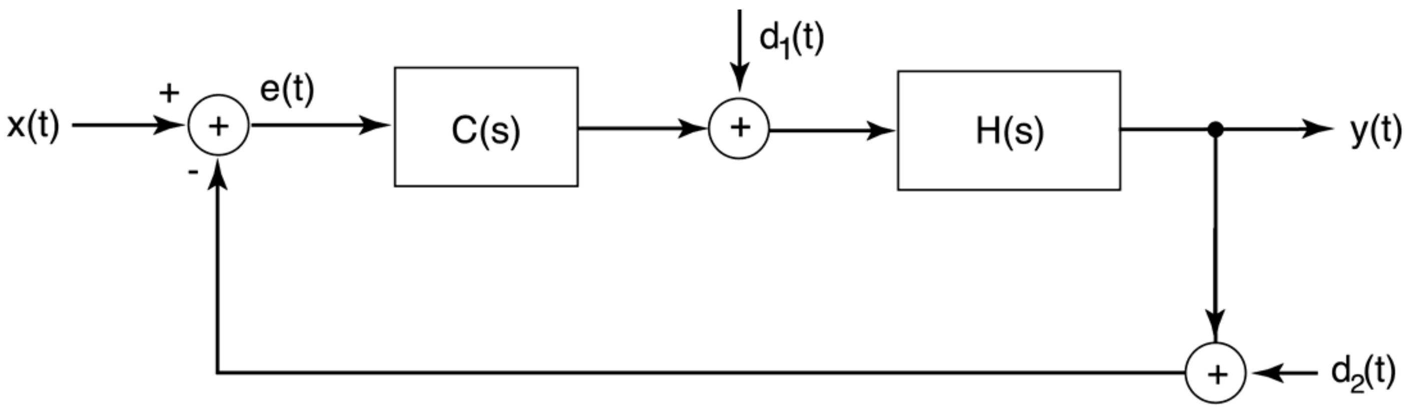
Basic example: Tracking error for a step input

Suppose $\quad x(t) = u(t) \longleftrightarrow X(s) = \frac{1}{s}$

Then $\quad \lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1}{1 + C(s)P(s)}$

Disturbance Rejection

There may be *other* objectives in feedback controls due to unavoidable disturbances.

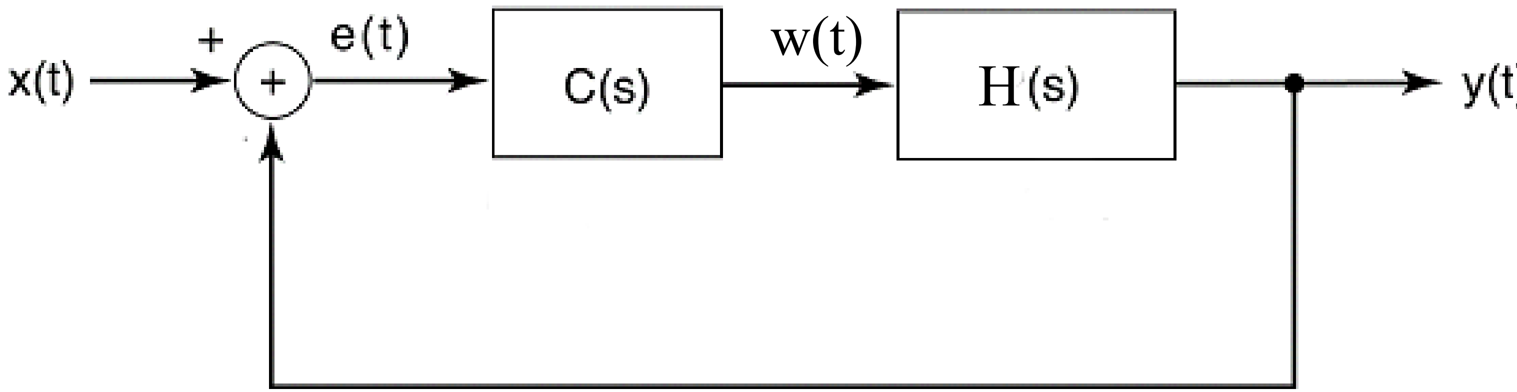


$$E(s) = \left[\frac{1}{1+C(s)H(s)}\right] X(s) - \left[\frac{H(s)}{1+C(s)H(s)}\right] D_1(s) - \left[\frac{1}{1+C(s)H(s)}\right] D_2(s)$$

Clearly, sensitivities to the disturbances $D_1(s)$ and $D_2(s)$ are much reduced when the amplitude of the loop gain

$$|C(s)H(s)| >> 1$$

Internal Instabilities Due to Pole-Zero Cancellation



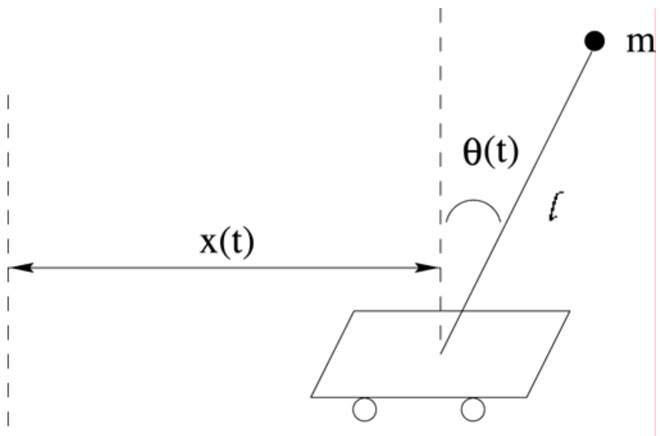
$$C(s) = \frac{1}{s(s+1)} \quad , \quad H(s) = \frac{s}{s+2}$$

$$Y(s) = \frac{C(s)H(s)}{1+C(s)H(s)}X(s) = \frac{1}{\underbrace{s^2+3s+3}_{\text{Stable}}}X(s)$$

However

$$W(s) = \frac{C(s)}{1+C(s)H(s)}X(s) = \frac{s+2}{\underbrace{s(s^2+3s+3)}_{\text{Unstable}}}X(s)$$

Inverted Pendulum



m = Mass, $\quad I$ = moment of inertia

$$mgl\sin\theta(t) - ml\frac{d^2x(t)}{dt^2}\cos\theta(t) = I\frac{d^2\theta(t)}{dt^2} \qquad \text{(Moment balance)}$$

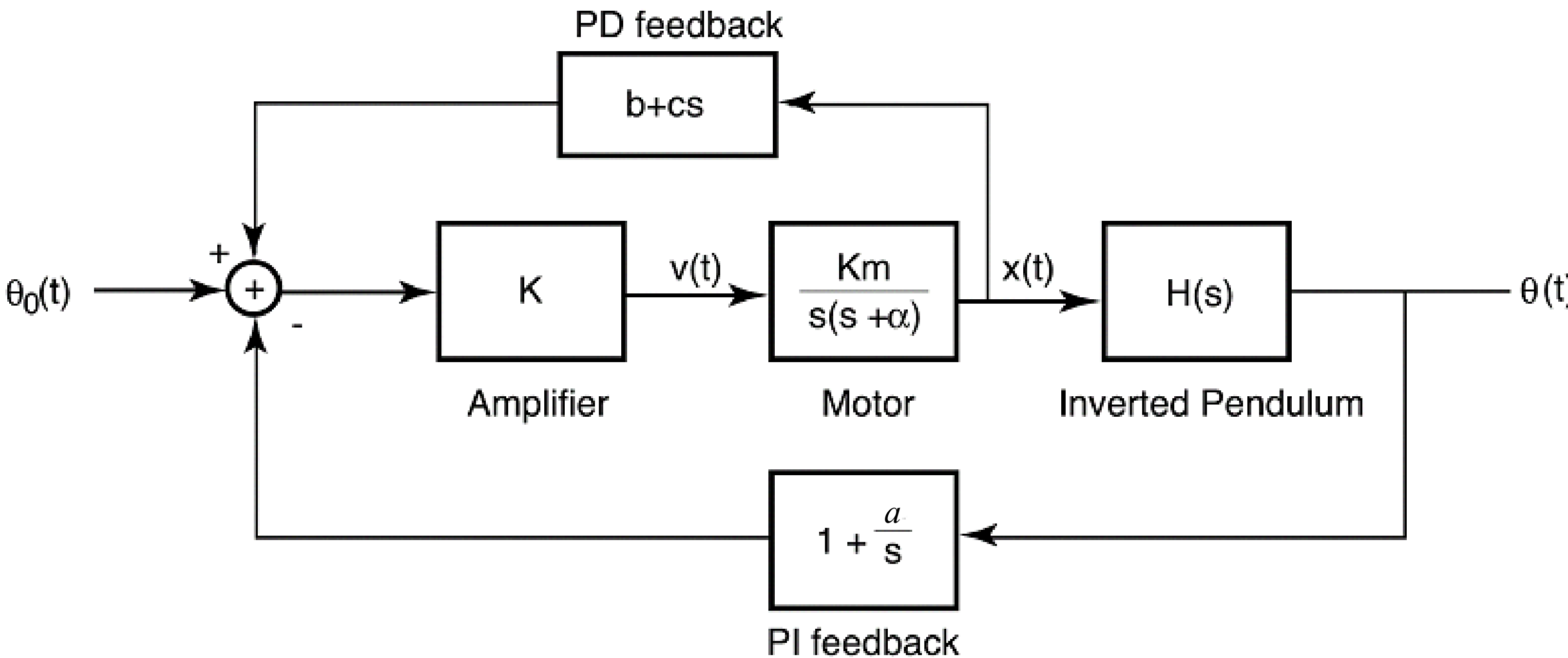
Linearize assuming θ is small: $\sin\theta \approx \theta, \cos\theta \approx 1$

$$I\frac{d^2\theta(t)}{dt^2} - mgl\theta(t) = -ml\frac{d^2x(t)}{dt^2}$$

$$\Theta(s) = \underbrace{\frac{-mls^2}{Is^2 - mgl}}_{H(s)} X(s)$$

— Unstable!

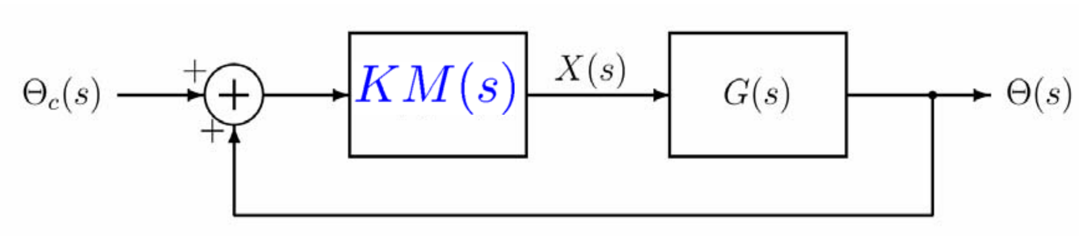
Feedback System to Stabilize the Pendulum



- PI feedback stabilizes θ
- Subtle problem: internal instability in x(t)!
  - Additional PD feedback around motor / amplifier centers the pendulum

Root Locus & the Inverted Pendulum

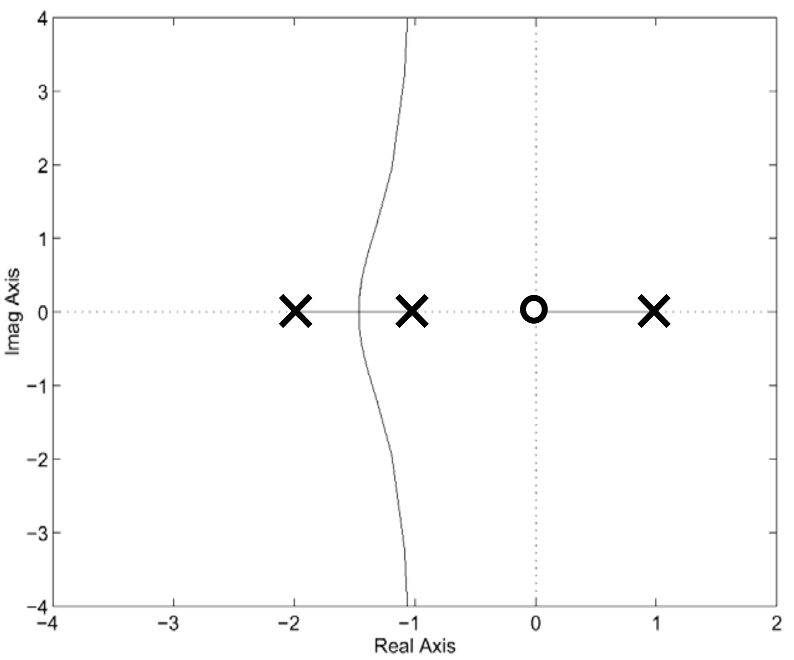
- Attempt #1: Negative feedback driving the motor



$$G(s) = \frac{\Theta(s)}{X(s)} = \frac{-s^2/g}{(\tau_L s + 1)(\tau_L s - 1)}$$

$$M(s) = \frac{X(s)}{V(s)} = \frac{k_m}{s(\tau_M s + 1)}$$

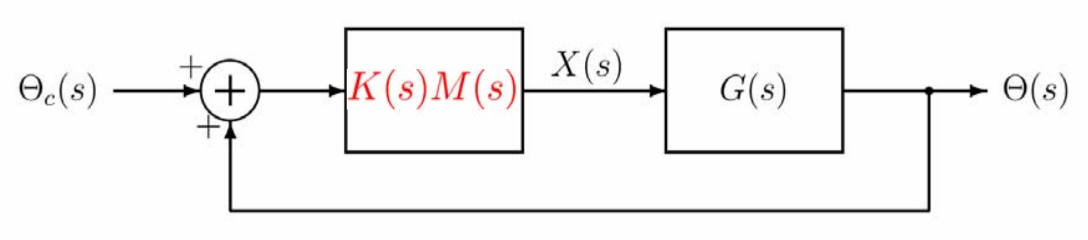
- Root locus of *M(s)G(s)*
  - Remains unstable!



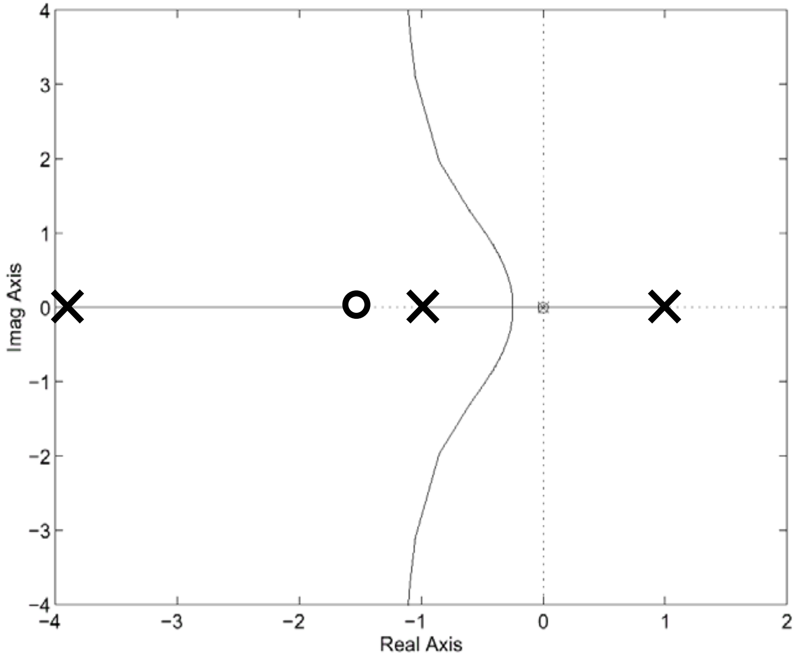
after K. Lundberg

Root Locus & the Inverted Pendulum

- Attempt #2: Proportional/Integral Compensator



$$K(s) = 1 + \frac{1}{\tau_k s}$$



- Root locus of *K(s)M(s)G(s)*
  - Stable for large enough K

after K. Lundberg

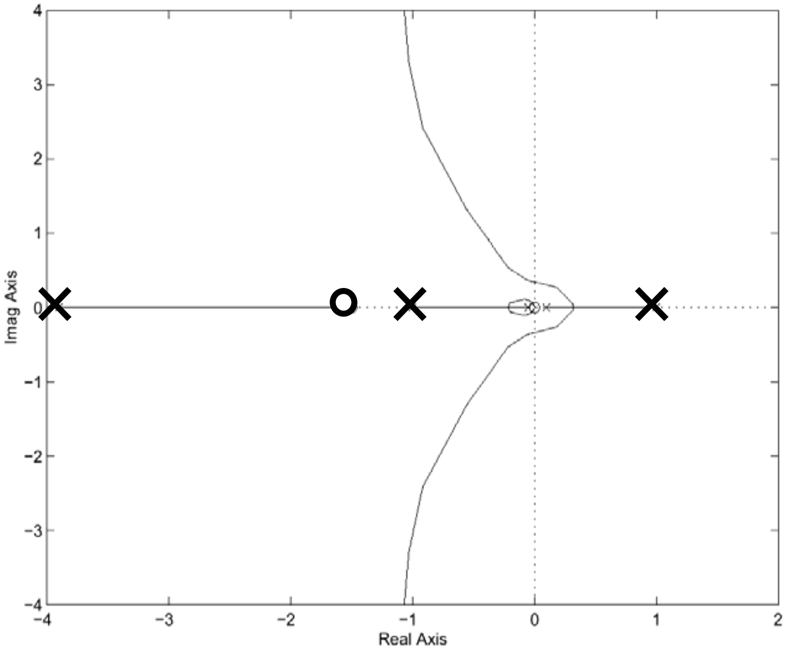
Root Locus & the Inverted Pendulum

- BUT – x(t) unstable:

$$\frac{X(s)}{\Theta_c(x)} = \frac{K(s)M(s)}{1 - K(s)M(s)G(s)}$$

$$= \frac{1}{s^2}\left(\frac{k_M(\tau_K s + 1)(\tau_L^2 s^2 - 1)}{\tau_K(\tau_M s + 1)(\tau_L^2 s^2 - 1) + (k_M/g)(\tau_K s + 1)}\right)$$

System subject to drift...

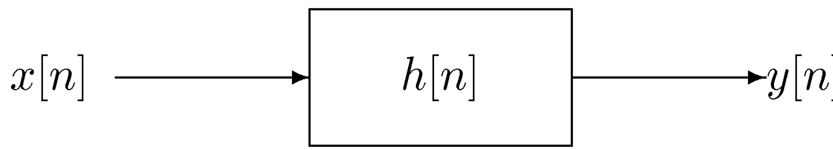


- Solution: add PD feedback around motor and compensator:

after K. Lundberg

The z-Transform

Motivation: Analogous to Laplace Transform in CT



We now do *not* restrict ourselves just to $z = e^{j\omega}$

$$x[n] = \underbrace{z^n}_{\substack{\text{Eigenfunction} \\ \text{for DT LTI}}} \longrightarrow y[n] = H(z)z^n$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \quad \text{assuming it converges}$$

The (Bilateral) z-Transform

$$x[n] \longleftrightarrow X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \mathcal{Z}\{x[n]\}$$

The ROC and the Relation Between *z*T and DTFT

$$z = re^{j\omega} \quad , r = |z|$$

$$\begin{aligned} X(re^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n] \left(re^{j\omega}\right)^{-n} = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n} \\ &= \mathcal{F}\{x[n]r^{-n}\} \end{aligned}$$

- $\text{ROC} = \left\{ z = re^{j\omega} \text{ at which } \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty \right\}$

  — depends only on $r = |z|$, just like the ROC in s-plane only depends on $Re(s)$

- Unit circle ($r = 1$) in the ROC $\Rightarrow$ DTFT $X(e^{j\omega})$ exists

Example #1

$x[n] = a^n u[n]$ - right-sided

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

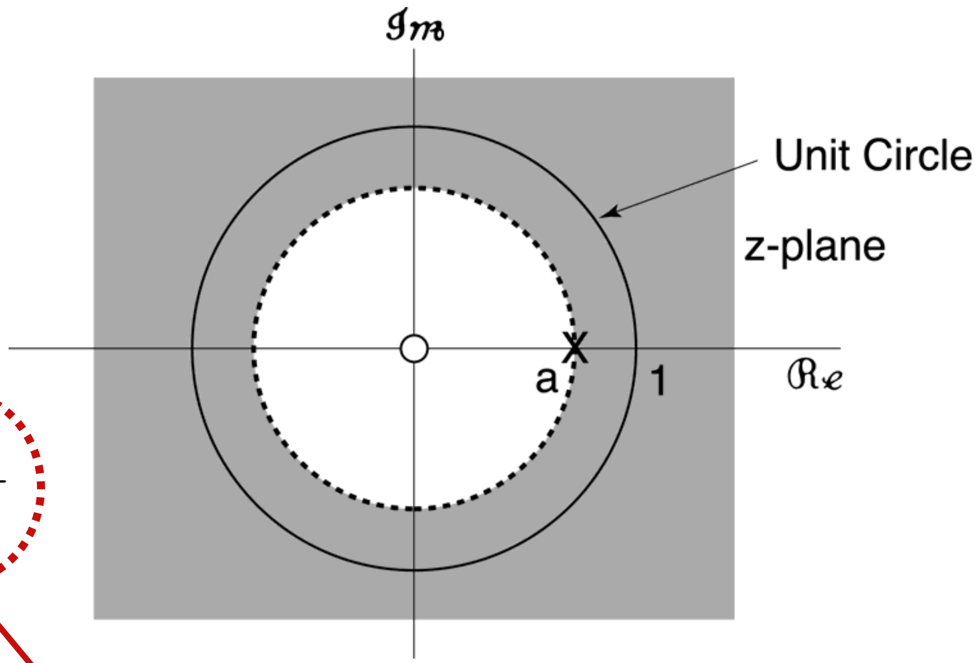
This form for PFE and inverse z-transform

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

This form to find pole and zero locations

If $|az^{-1}| < 1$, i.e., $|z| > |a|$

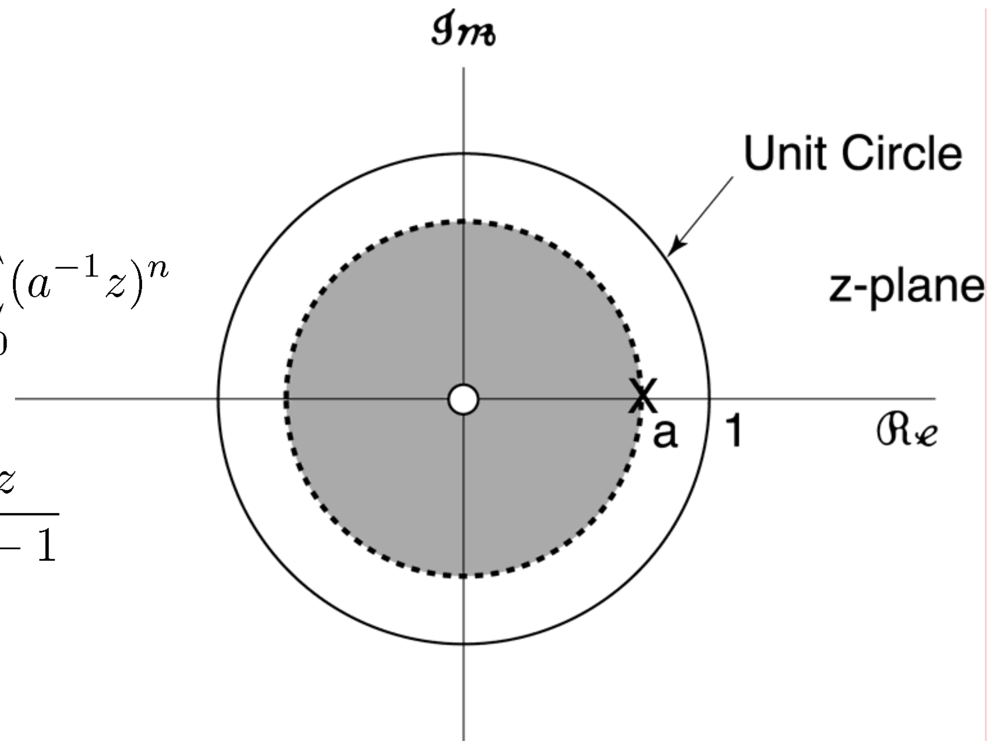
That is, ROC $|z| > |a|$, outside a circle



Example #2:

$x[n] = -a^n u[-n-1]$ - left-sided

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \{-a^n u[-n-1] z^{-n}\} \\
&= -\sum_{n=-\infty}^{-1} a^n z^{-n} \\
&= -\sum_{n=1}^{\infty} a^{-n} z^n = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n \\
&= 1 - \frac{1}{1 - a^{-1} z} = \frac{a^{-1} z}{a^{-1} z - 1} \\
&= \frac{z}{z-a},
\end{aligned}
$$



If $|a^{-1} z| < 1$, $i.e.$, $|z| < |a|$

Same $X(z)$ as in **Ex #1**, but different ROC.

Rational z-Transforms

$x[n]$ = linear combination of exponentials for $n > 0$ and for $n < 0$

$$\Downarrow$$

$$X(z) \text{ is rational}$$

$$X(z) = \frac{N(z)}{D(z)}$$

$N(z)$, $D(z)$: Polynomials in z

— characterized (except for a gain) by its poles and zeros