Signals and Systems

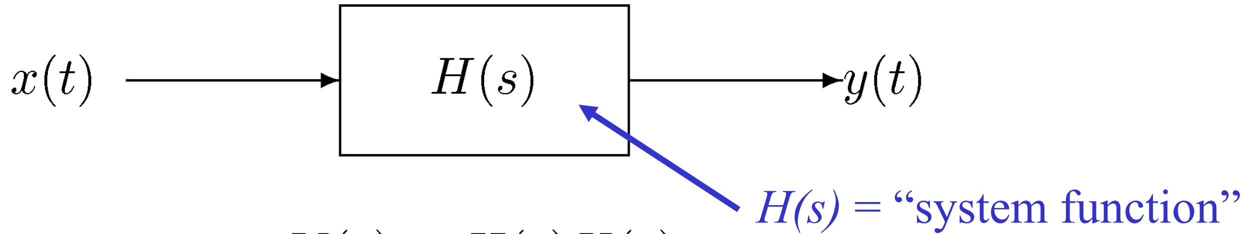
Fall 2003

Lecture #19

18 November 2003

1. CT System Function Properties
2. System Function Algebra and Block Diagrams
3. Unilateral Laplace Transform and Applications

CT System Function Properties



$$Y(s) = H(s)X(s)$$

1) System is stable $\Leftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty$ $\Leftrightarrow$ ROC of $H(s)$ includes $j\omega$ axis

2) Causality $\Rightarrow$ $h(t)$ right-sided signal $\Rightarrow$ ROC of $H(s)$ is a right-half plane

Question:

If the ROC of $H(s)$ is a right-half plane, is the system causal?

Ex.

$$\begin{aligned} H(s) &= \frac{e^{sT}}{s+1}, \quad \Re e\{s\} > -1 \Rightarrow h(t) \text{ right-sided} \\ h(t) &= \mathcal{L}^{-1}\left\{\frac{e^{sT}}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}_{t \to t+T} = e^{-t}u(t)|_{t \to t+T} \\ &= e^{-(t+T)}u(t+T) \neq 0 \quad \text{at} \quad t < 0 \end{aligned}$$

Non-causal

Properties of CT Rational System Functions

a) However, if $H(s)$ is ***rational,*** then

The system is causal $\Leftrightarrow$ The ROC of $H(s)$ is to the right of the rightmost pole

b) If $H(s)$ is rational and is the system function of a causal system, then

The system is stable $\Leftrightarrow$ $j\omega$-axis is in ROC $\Leftrightarrow$ all poles are in LHP

Checking if All Poles Are In the Left-Half Plane

$$H(s) = \frac{N(s)}{D(s)}$$

Poles are the roots of $D(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$

Method #1: Calculate all the roots and see!
Method #2: Routh-Hurwitz – Without having to solve for roots.

	Polynomial	Condition so that all roots are in the LHP
First-order	$s + a_0$	$a_0 > 0$
Second-order	$s^2 + a_1 s + a_0$	$a_1 > 0, a_0 > 0$
Third-order	$s^3 + a_2 s^2 + a_1 s + a_0$	$a_2 > 0, a_1 > 0, a_0 > 0$ <u>and</u> $a_0 < a_1 a_2$
	$\vdots$	$\vdots$

Initial- and Final-Value Theorems

If $x(t) = 0$ for $t < 0$ and there are no impulses or higher order discontinuities at the origin, then

$$x(0^+) = \lim_{s \to \infty} sX(s)$$

Initial value

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$, then

$$x(\infty) = \lim_{s \to 0} sX(s)$$

Final value

Applications of the Initial- and Final-Value Theorem

For

$$X(s) = \frac{N(s)}{D(s)}$$

n - order of polynomial $N(s)$, d - order of polynomial $D(s)$

- Initial value:

$$x(0^+) = \lim_{s \to \infty} sX(s) = \begin{cases} 0 & d > n+1 \\ \text{finite } \neq 0 & d = n+1 \\ \infty & d < n+1 \end{cases}$$

E.g. $X(s) = \dfrac{1}{s+1}$ $\quad x(0^+) = ?$

- Final value

$$\text{If } x(\infty) = \lim_{s \to 0} sX(s) = 0 \Rightarrow \lim_{s \to 0} X(s) < \infty$$

$\Rightarrow$ No poles at $s = 0$

LTI Systems Described by LCCDEs

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Repeated use of differentiation property: $\frac{d}{dt} \leftrightarrow s, \quad \frac{d^k}{dt^k} \leftrightarrow s^k$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\Downarrow$$

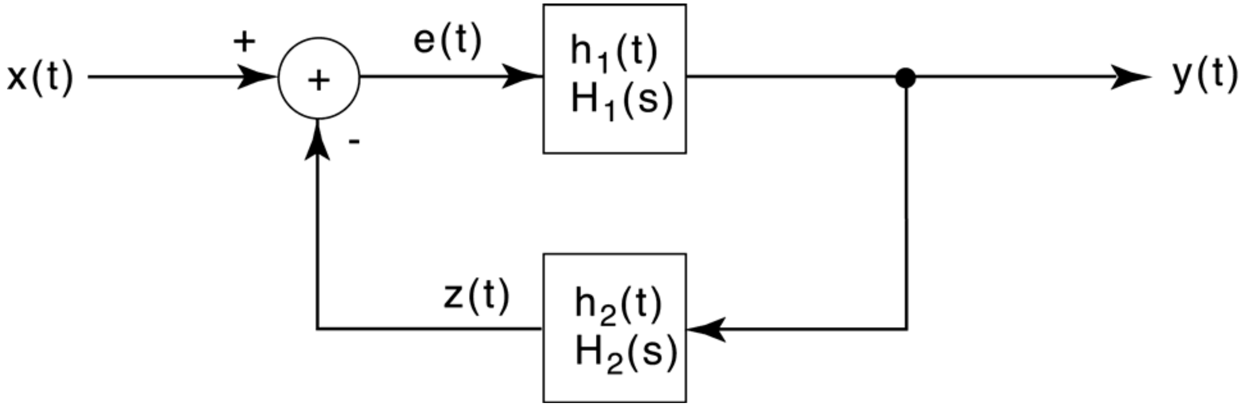
$$Y(s) = H(s)X(s)$$

where $H(s) = \underbrace{\frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}}_{\text{Rational}}$ $\begin{matrix} \longleftarrow \\ \longleftarrow \end{matrix}$ roots of numerator ⇒ *zeros*; roots of denominator ⇒ *poles*

ROC =? Depends on:
1) Locations of *all* poles.
2) Boundary conditions, *i.e.* right-, left-, two-sided signals.

System Function Algebra

Example: A basic feedback system consisting of ***causal*** blocks



$$E(s) = X(s) - Z(s) = X(s) - H_2(s)Y(s)$$
$$Y(s) = H_1(s)E(s) = H_1(s)[X(s) - H_2(s)Y(s)]$$

$$\Downarrow$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

← More on this later in feedback

ROC: Determined by the roots of $1+H_1(s)H_2(s)$, instead of $H_1(s)$

Block Diagram for Causal LTI Systems with Rational System Functions

Example:

$$Y(s) = H(s)X(s)$$

$$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s + 2} = \left(\frac{1}{s^2 + 3s + 2}\right)(2s^2 + 4s - 6)$$

— Can be viewed as cascade of two systems.

Define:

$$W(s) = \frac{1}{s^2 + 3s + 2}X(s)$$

$$\frac{d^2w(t)}{dt^2} + 3\frac{dw(t)}{dt} + 2w(t) = x(t), \quad \text{initially at rest}$$

$$\text{or} \quad \frac{d^2w(t)}{dt^2} = x(t) - 3\frac{dw(t)}{dt} - 2w(t)$$

Similarly

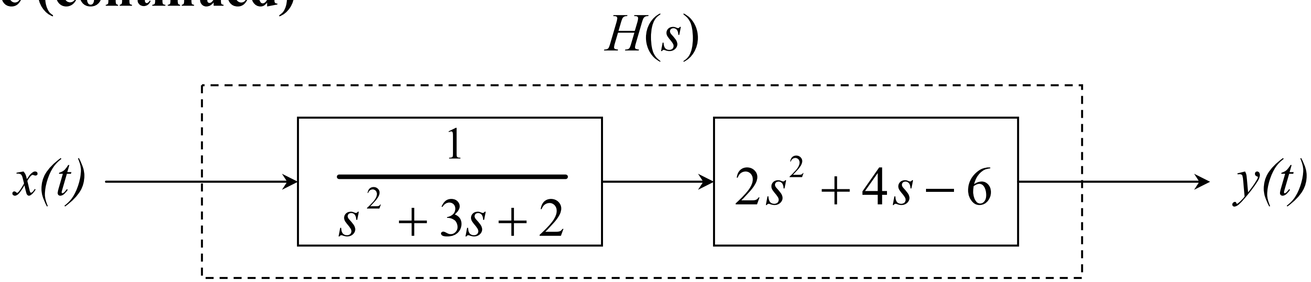
$$Y(s) = (2s^2 + 4s - 6)W(s)$$

$$\Downarrow$$

$$y(t) = 2\frac{d^2w(t)}{dt^2} + 4\frac{dw(t)}{dt} - 6w(t)$$

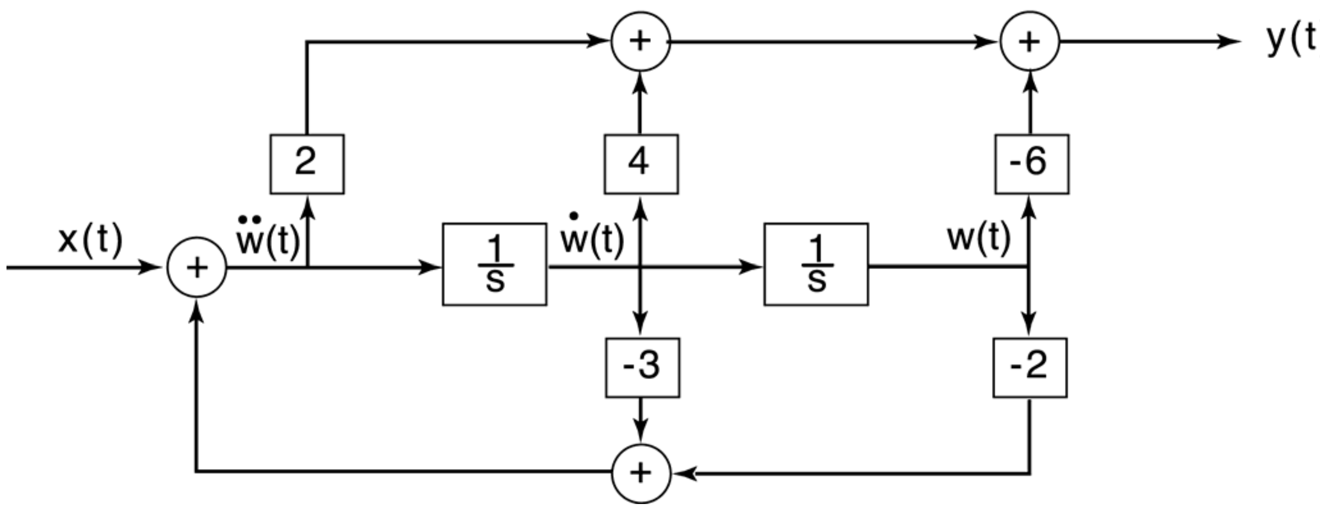
Example (continued)

Instead of



We can construct *H*(*s*) using:

$$\begin{aligned} \frac{d^2w(t)}{dt^2} &= x(t) - 3\frac{dw(t)}{dt} - 2w(t) \\ y(t) &= 2\frac{d^2w(t)}{dt^2} + 4\frac{dw(t)}{dt} - 6w(t) \end{aligned}$$

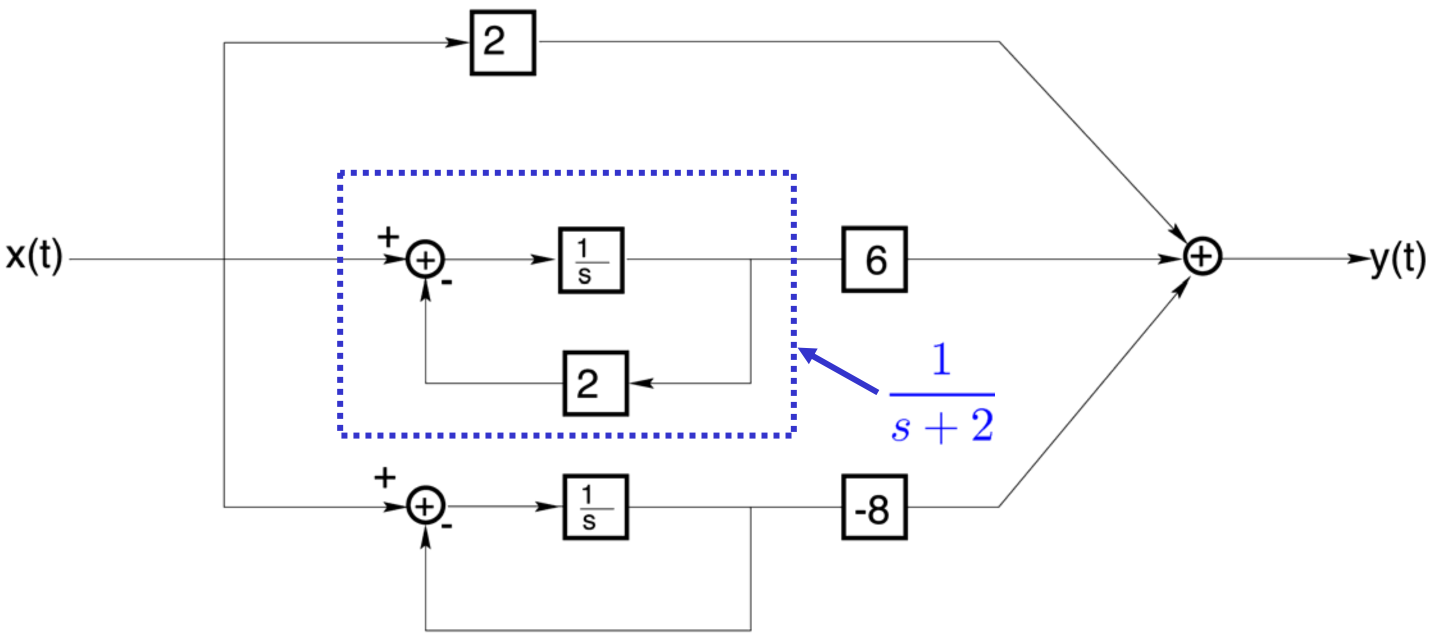


Notation: 1/*s* — an integrator

Note also that

$$H(s) = \left[\frac{2(s-1)}{s+2}\right]\left[\frac{s+3}{s+1}\right] = \left[\frac{s+3}{s+2}\right]\left[\frac{2(s-1)}{s+1}\right] \quad - \text{ Cascade}$$

$$\overset{PFE}{=} \quad 2 + \frac{6}{s+2} - \frac{8}{s+1} \quad - \text{ parallel connection}$$



Lesson to be learned: There are many *different* ways to construct a system that performs a certain function.

The Unilateral Laplace Transform

(The preferred tool to analyze causal CT systems described by LCCDEs with **initial conditions**)

$$\mathcal{X}(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt = \mathcal{UL}\{x(t)\}$$

Note:

1) If $x(t) = 0$ for $t < 0$, then $X(s) = \mathcal{X}(s)$

2) Unilateral LT of $x(t)$ = Bilateral LT of $x(t)u(t\text{-})$

3) For example, if $h(t)$ is the impulse response of a causal LTI system, then

$$H(s) = \mathcal{H}(s)$$

4) Convolution property: If $x_1(t) = x_2(t) = 0$ for $t < 0$, then

$$\boxed{\mathcal{UL}\{x_1(t) * x_2(t)\} = \mathcal{X}_1(x)\mathcal{X}_2(s)}$$

Same as Bilateral Laplace transform

Differentiation Property for Unilateral Laplace Transform

$$x(t) \longleftrightarrow \mathcal{X}(s)$$

$$\Downarrow$$

$$\frac{dx(t)}{dt} \longleftrightarrow s\mathcal{X}(s) - x(0^-)$$

Initial condition!

Derivation:

integration by parts

$\int f \cdot dg = fg - \int g \cdot df$

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt \quad = \quad s \underbrace{\int_{0^-}^{\infty} x(t) e^{-st} dt}_{\mathcal{X}(s)} \quad + \quad x(t)e^{-st}\Big|_{0-}^{\infty}$$

$$= \quad s\mathcal{X}(s) - x(0^-)$$

Note:

$$\frac{d^2x(t)}{dt^2} = \frac{d}{dt}\left\{\frac{dx(t)}{dt}\right\} \quad \longleftrightarrow \quad s(\overbrace{s\mathcal{X}(s) - x(0^-)}^{\mathcal{UL}\frac{dx(t)}{dt}}) - x'(0^-)$$

$$\longleftrightarrow \quad s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$

Use of ULTs to Solve Differentiation Equations with Initial Conditions

Example:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$y(0^-) = \beta\,,\; y'(0^-) = \gamma\,,\; x(t) = \alpha u(t)$$

Take ULT:

$$\underbrace{s^2\mathcal{Y}(s) - \beta s - \gamma}_{\mathcal{UL}\left\{\frac{d^2y}{dt^2}\right\}} + 3\underbrace{(\mathcal{Y}(s) - \beta)}_{\mathcal{UL}\left\{\frac{dy}{dt}\right\}} + 2\mathcal{Y}(s) = \frac{\alpha}{s}$$

$$\Downarrow$$

$$\mathcal{Y}(s) = \underbrace{\frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}}_{ZIR} + \underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{ZSR}$$

ZIR — Response for zero input $x(t)=0$

ZSR — Response for zero state, $\beta=\gamma=0$, initially at rest

Example (continued)

- Response for LTI system initially at rest ($\beta = \gamma = 0$)

$$\Downarrow$$

$$\mathcal{H}(s) = \frac{\mathcal{Y}(s)}{\mathcal{X}(s)} = \frac{1}{(s+1)(s+2)} = H(s)$$

- Response to initial conditions alone ($\alpha = 0$).
  For example:

$$x(t) = 0\,(\text{no input}), \quad y(0^-) = 1, \quad y'(0^-) = 0 \quad (\beta = 1, \gamma = 0)$$

$$\Downarrow$$

$$\mathcal{Y}(s) = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$

$$\Downarrow$$

$$y(t) = 2e^{-t} - e^{-2t}, \quad t \geq 0$$