Signals and Systems

Fall 2003

Lecture #18

6 November 2003

- Inverse Laplace Transforms
- Laplace Transform Properties
- The System Function of an LTI System
- Geometric Evaluation of Laplace Transforms and Frequency Responses

Inverse Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt, \quad s = \sigma + j\omega \in \text{ ROC}$$
$$= \mathcal{F}\{x(t)e^{-\sigma t}\}$$

Fix $\sigma \in$ ROC and apply the inverse Fourier transform

$$x(t)e^{-\sigma t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{j\omega t}d\omega$$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t}d\omega$$

But $s = \sigma + j\omega$ (σ fixed) $\Rightarrow ds = jd\omega$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\omega}^{\sigma + j\omega} X(s)e^{st}ds$$

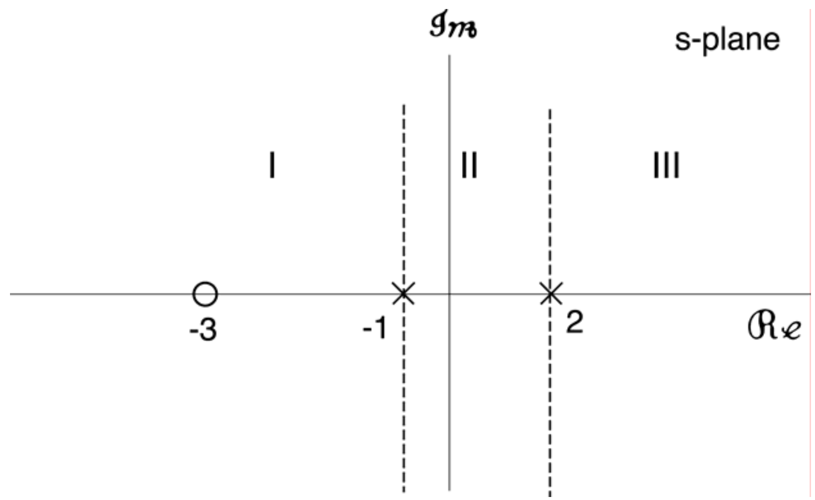
Inverse Laplace Transforms Via Partial Fraction Expansion and Properties

Example:

$$X(s) = \frac{s+3}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$A = -\frac{2}{3}, \quad B = \frac{5}{3}$$

Three possible ROC's — corresponding to three *different* signals



Recall $\quad \frac{1}{s+a}, \quad \Re e\{s\} < -a \longleftrightarrow -e^{-at}u(-t)$ left-sided

$\frac{1}{s+a}, \quad \Re e\{s\} > -a \longleftrightarrow e^{-at}u(t)$ right-sided

ROC I: — Left-sided signal.

$$\begin{aligned} x(t) &= -Ae^{-t}u(-t) - Be^{2t}u(-t) \\ &= \left[\frac{2}{3}e^{-t} - \frac{5}{3}e^{2t}\right]u(-t) \qquad \text{Diverges as } t \to -\infty \end{aligned}$$

ROC II: — Two-sided signal, has Fourier Transform.

$$\begin{aligned} x(t) &= Ae^{-t}u(t) - Be^{2t}u(-t) \\ &= -\left[\frac{2}{3}e^{-t}u(t) + \frac{5}{3}e^{2t}u(-t)\right] \quad \to 0 \text{ as } t \to \pm\infty \end{aligned}$$

ROC III:— Right-sided signal.

$$\begin{aligned} x(t) &= Ae^{-t}u(t) + Be^{2t}u(t) \\ &= \left[-\frac{2}{3}e^{-t} + \frac{5}{3}e^{2t}\right]u(t) \qquad \text{Diverges as } t \to +\infty \end{aligned}$$

Properties of Laplace Transforms

- Many parallel properties of the CTFT, but for Laplace transforms we need to determine implications for the ROC

- For example:

<u>Linearity</u>

$$ax_1(t) + bx_2(t) \longleftrightarrow aX_1(s) + bX_2(s)$$

ROC at least the intersection of ROCs of $X_1(s)$ and $X_2(s)$

ROC can be bigger (due to pole-zero cancellation)

E.g. $\quad x_1(t) = x_2(t)$ and $a = -b$

Then $\quad ax_1(t) + bx_2(t) = 0 \longrightarrow X(s) = 0$

$\Rightarrow$ ROC entire s-plane

Time Shift

$$x(t-T) \longleftrightarrow e^{-sT}X(s), \text{ same ROC as } X(s)$$

Example:

$$\frac{e^{3s}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad ?$$

$$\frac{e^{-sT}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad e^{-2t}u(t)|_{t \to t-T}$$

$$\downarrow T = -3$$

$$\frac{e^{3s}}{s+2}, \quad \Re e\{s\} > -2 \quad \longleftrightarrow \quad e^{-2(t+3)}u(t+3)$$

Time-Domain Differentiation

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} X(s)e^{st}ds, \quad \frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} sX(s)e^{st}ds$$

$$\Downarrow$$

$$\frac{dx(t)}{dt} \longleftrightarrow sX(s), \text{ with ROC containing the ROC of } X(s)$$

ROC could be bigger than the ROC of $X(s)$, if there is pole-zero cancellation. E.g.,

$$\begin{array}{rcccll} x(t) & = & u(t) & \leftrightarrow & \frac{1}{s}, & \Re e\{s\} > 0 \\ \frac{dx(t)}{dt} & = & \delta(t) & \leftrightarrow & 1 = s \cdot \frac{1}{s} & \text{ROC} = \text{entire s-plane} \end{array}$$

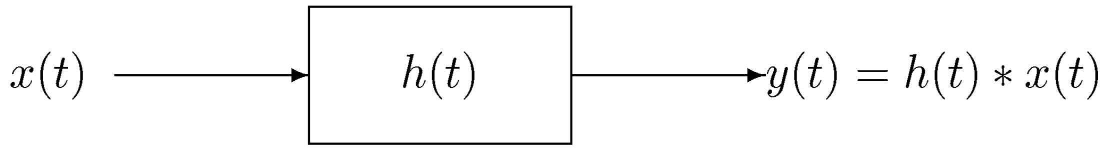
s-Domain Differentiation

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds}, \text{ with same ROC as } X(s)$$

(Derivation is similar to $\frac{d}{dt} \leftrightarrow s$)

E.g., $$te^{-at}u(t) \leftrightarrow -\frac{d}{ds}\left[\frac{1}{s+a}\right] = \frac{1}{(s+a)^2}, \quad \Re e\{s\} > -a$$

Convolution Property



For $\quad x(t) \longleftrightarrow X(s), y(t) \longleftrightarrow Y(s), h(t) \longleftrightarrow H(s)$

Then

$$Y(s) = H(s) \cdot X(s)$$

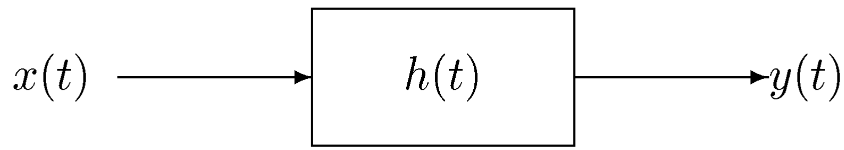
- ROC of $Y(s) = H(s)X(s)$: at least the overlap of the ROCs of $H(s)$ & $X(s)$
- ROC could be empty if there is no overlap between the two ROCs
  E.g.

$$x(t) = e^{t}u(t), \text{ and } \quad h(t) = -e^{-t}u(-t)$$

- ROC could be larger than the overlap of the two. E.g.

$$x(t) * h(t) = \delta(t)$$

The System Function of an LTI System



$h(t) \quad \longleftrightarrow \quad H(s)$ – the system function

The system function characterizes the system

$$\Downarrow$$

System properties correspond to properties of $H(s)$ and its ROC

A first example:

$$\text{System is stable} \Leftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty \quad \Leftrightarrow \quad \begin{array}{l}\text{ROC of } H(s) \\ \text{includes the } j\omega \text{ axis}\end{array}$$

Geometric Evaluation of Rational Laplace Transforms

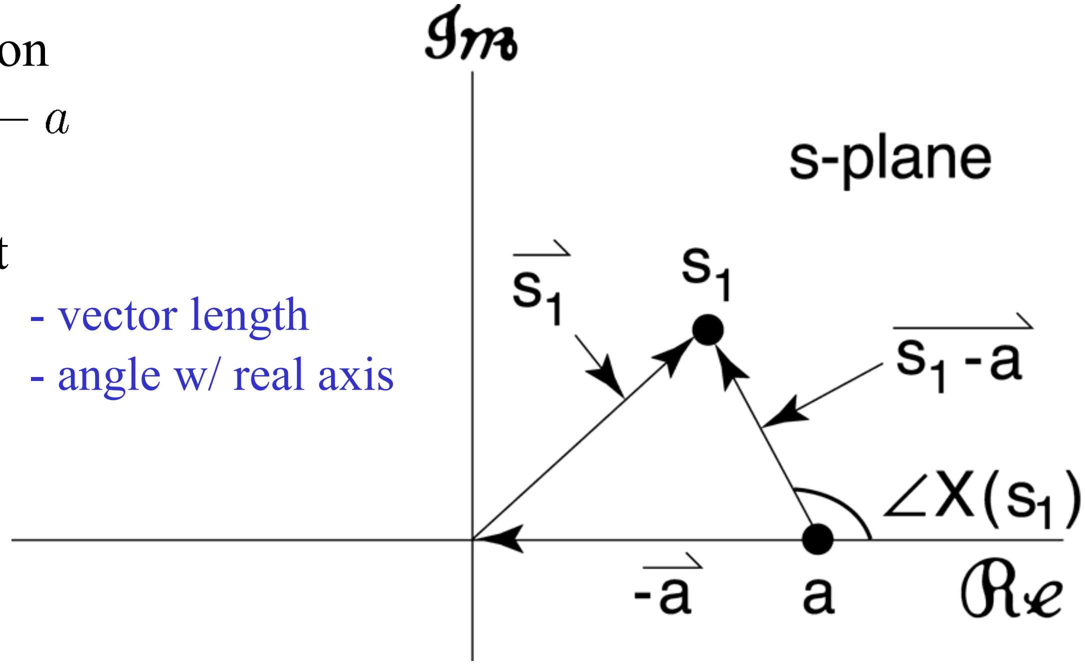
Example #1: $X_1(s) = s - a$ A first-order zero

Graphic evaluation of $X_1(s) = s_1 - a$

Can reason about

$|X_1(s)|$ - vector length

$\angle X_1(s)$ - angle w/ real axis



Example #2: A first-order pole

$$X_2(s) = \frac{1}{s-a} = \frac{1}{X_1(s)}$$

$$\Rightarrow |X_2(s)| = \frac{1}{|X_1(s)|} \quad (\text{or } \log|X_2(s)| = -\log|X_1(s)|)$$

$$\angle X_2(s) = -\angle X_1(s)$$

Still reason with vector, but remember to "invert" for poles

Example #3: A higher-order rational Laplace transform

$$X(s) = M\frac{\Pi_{i=1}^{R}(s-\beta_i)}{\Pi_{j=1}^{P}(s-\alpha_j)}$$

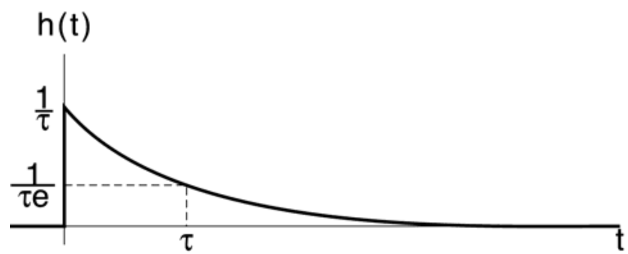
$$|X(s)| = |M|\frac{\Pi_{i=1}^{R}|s-\beta_i|}{\Pi_{j=1}^{P}|s-\alpha_j|}$$

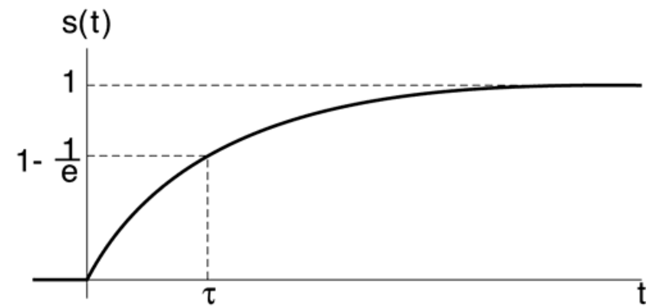
$$\angle X(s) = \angle M + \sum_{i=1}^{R}\angle(s-\beta_i) - \sum_{j=1}^{P}\angle(s-\alpha_j)$$

First-Order System

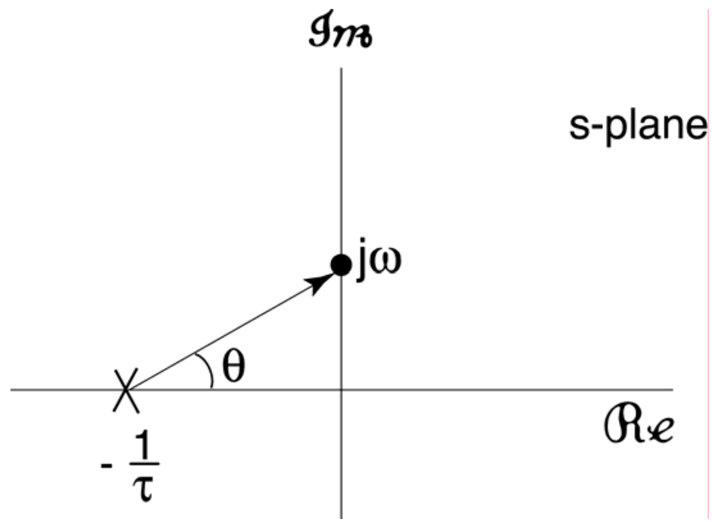
$$H(s) = \frac{1}{s\tau + 1} = \frac{1/\tau}{s + 1/\tau}, \Re e\{s\} > -\frac{1}{\tau}$$

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t) \qquad\qquad s(t) = [1 - e^{-t/\tau}]u(t)$$





Graphical evaluation of $H(j\omega)$: $H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau} = \frac{1}{\tau} \cdot \frac{1}{j\omega + 1/\tau}$

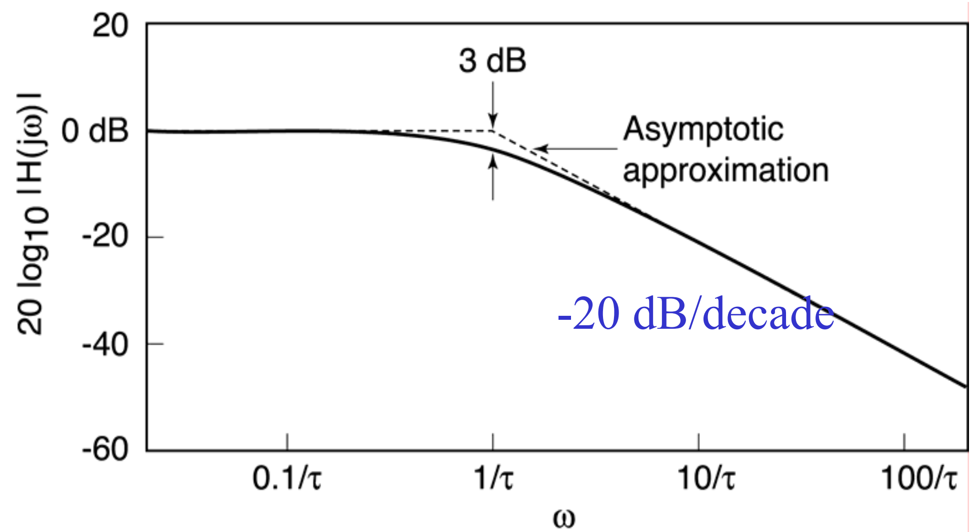


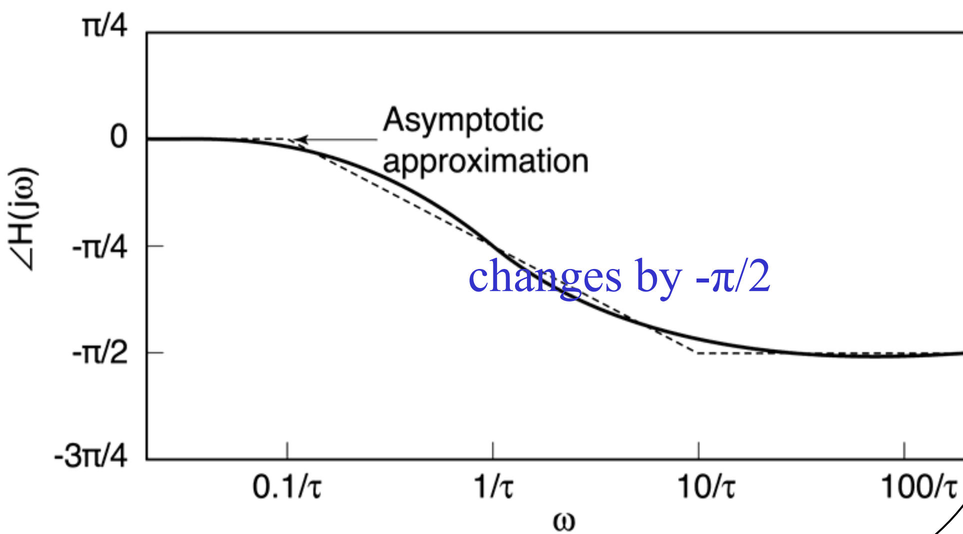
Bode Plot of the First-Order System

$$H(j\omega) = \frac{1/\tau}{j\omega + 1/\tau}$$

$$|H(j\omega)| = \frac{1/\tau}{\sqrt{\omega^2 + (1/\tau)^2}} = \begin{cases} 1 & \omega = 0 \\ 1/\sqrt{2} & \omega = 1/\tau \\ 1/\omega\tau & \omega >> 1/\tau \end{cases}$$

$$\angle H(j\omega) = -\theta = -\tan^{-1}(\omega\tau) = \begin{cases} 0 & \omega = 0 \\ -\pi/4 & \omega = 1/\tau \\ -\pi/2 & \omega >> 1/\tau \end{cases}$$





Second-Order System

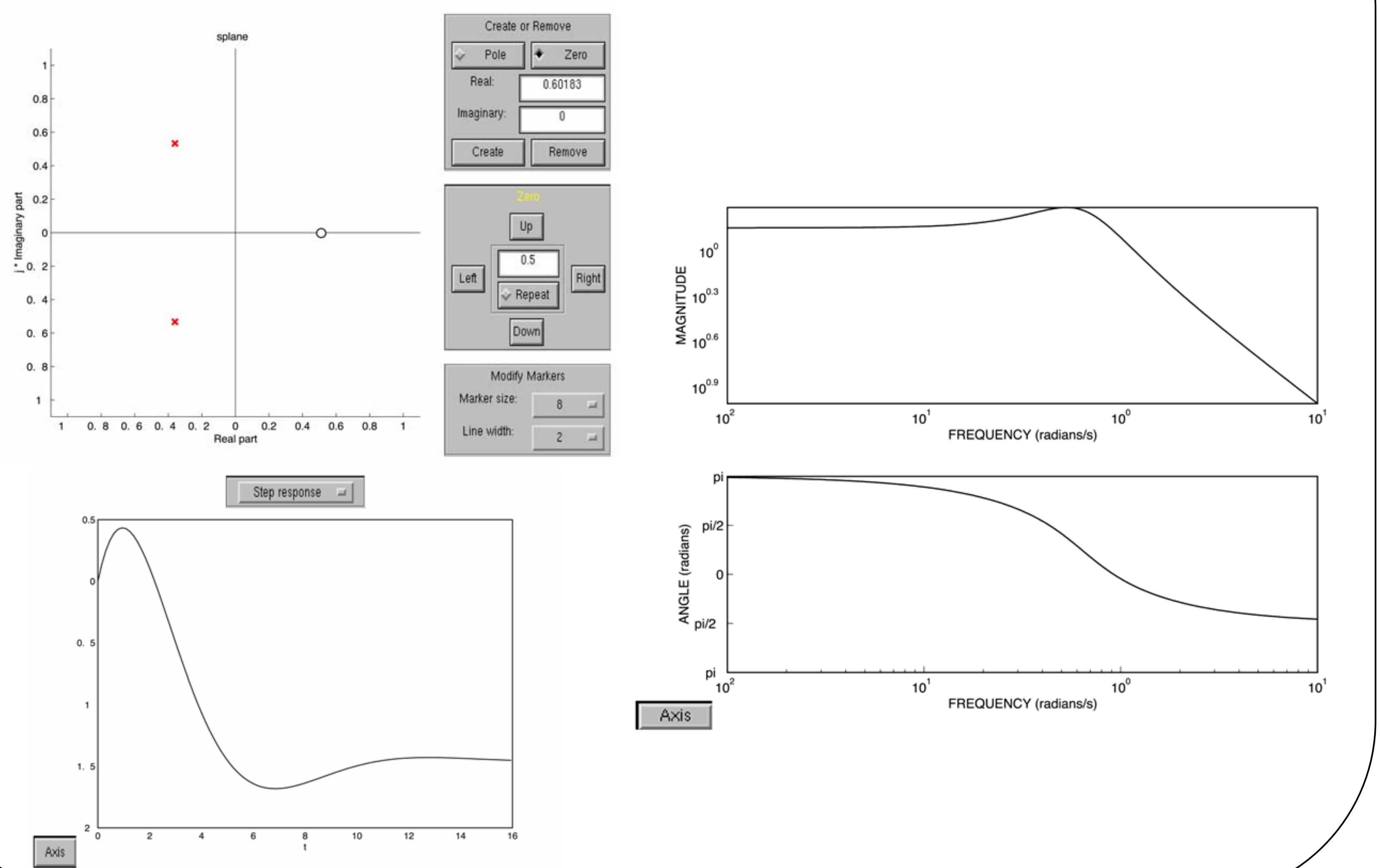
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad \text{ROC } \Re e\{s\} > \Re e(\text{pole})$$

$0 < \zeta < 1$ $\Rightarrow$ complex poles
— *Under*damped

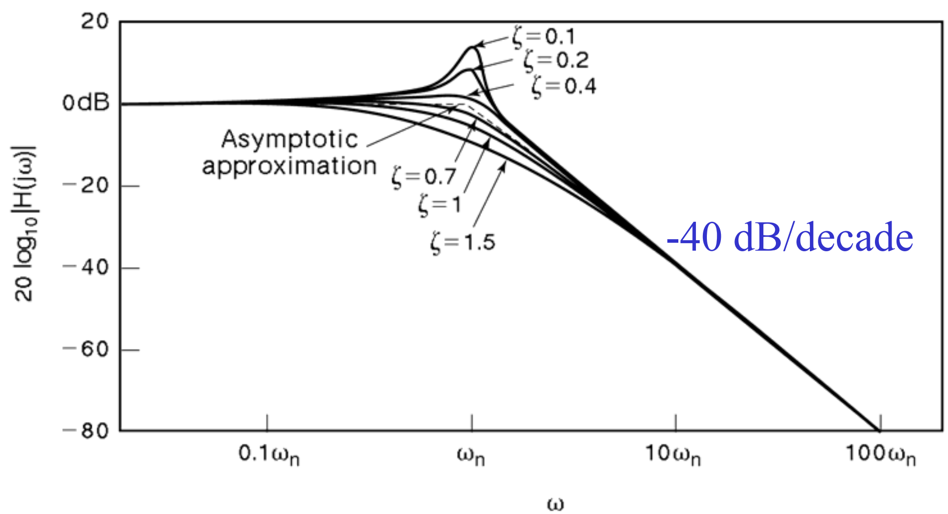
$\zeta = 1$ $\Rightarrow$ double pole at $s = -\omega_n$
— *Critically* damped

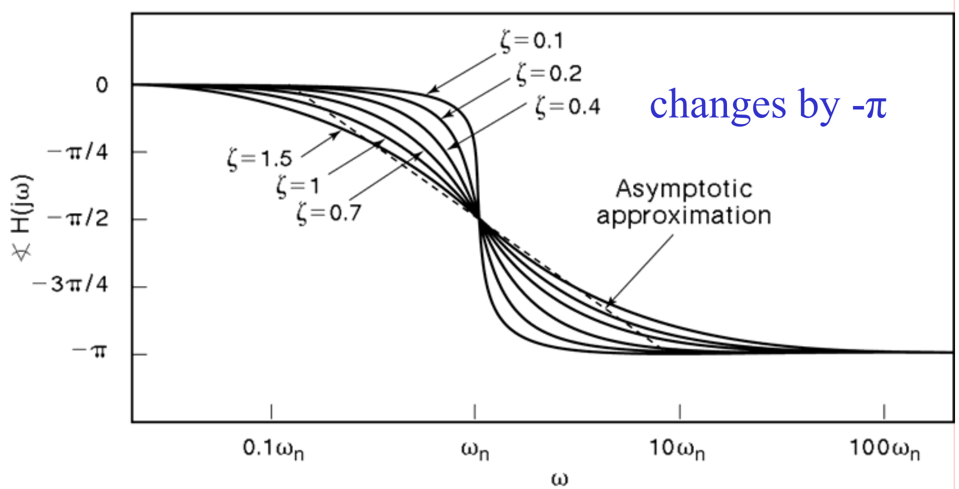
$\zeta > 1$ $\Rightarrow$ 2 poles on negative real axis
— *Over*damped

Demo Pole-zero diagrams, frequency response, and step response of first-order and second-order CT causal systems



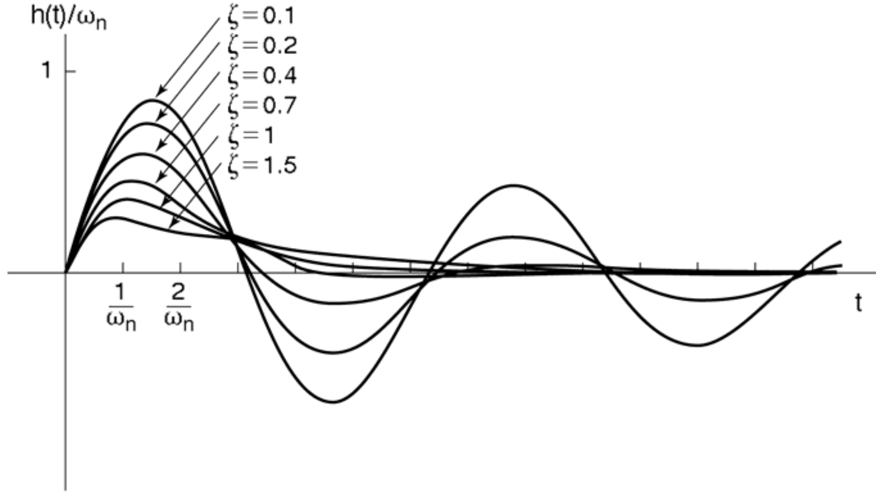
Bode Plot of a Second-Order System





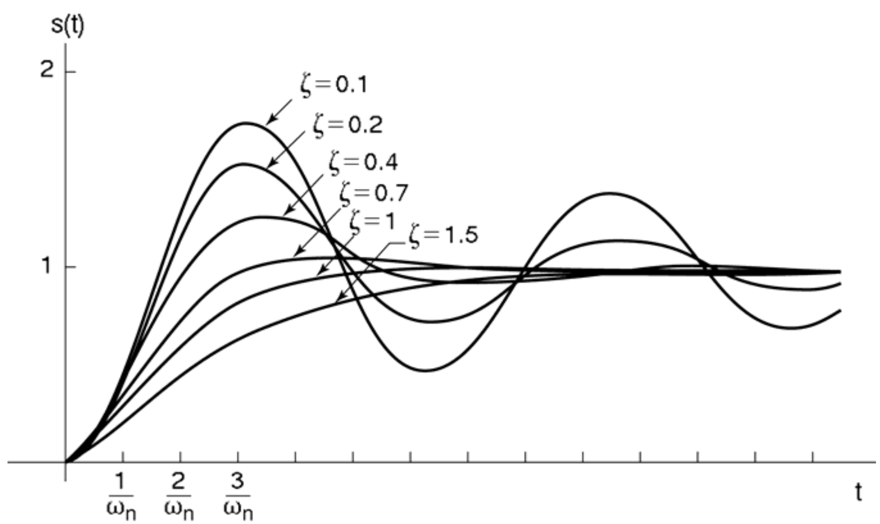
Top is flat when
$\zeta = 1/\sqrt{2} = 0.707$
$\Rightarrow$ a LPF for
$\omega < \omega_n$

Unit-Impulse and Unit-Step Response of a Second-Order System



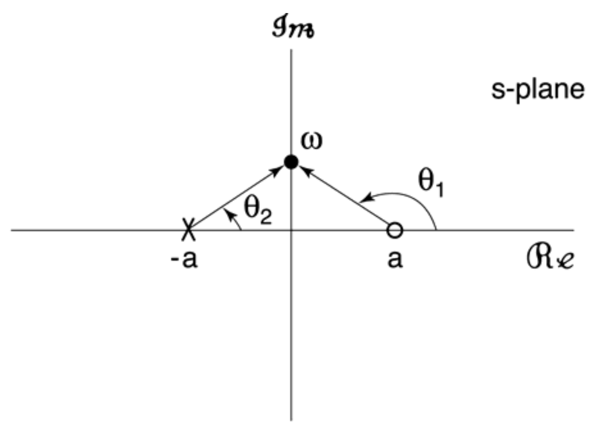
No oscillations when $\zeta \geq 1$

$\Rightarrow$ Critically (=) and over (>) damped.



First-Order All-Pass System

$$H(s) = \frac{s-a}{s+a}, \quad \Re e\{s\} > -a \quad (a > 0)$$



1. Two vectors have the same lengths
2. $$\begin{aligned} \angle H(j\omega) &= \theta_1 - \theta_2 \\ &= (\pi - \theta_2) - \theta_2 \\ &= \pi - 2\theta_2 \\ &= \begin{cases} \pi & \omega = 0 \\ \pi/2 & \omega = a \\ \backsim 0 & \omega >> a \end{cases} \end{aligned}$$