Signals and Systems

Fall 2003

Lecture #17

4 November 2003

1. Motivation and Definition of the (Bilateral) Laplace Transform
2. Examples of Laplace Transforms and Their Regions of Convergence (ROCs)
3. Properties of ROCs

Motivation for the Laplace Transform

- CT Fourier transform enables us to do a lot of things, e.g.
  - Analyze frequency response of LTI systems
  - Sampling
  - Modulation

    ⋮
- Why do we need yet another transform?

- One view of Laplace Transform is as an *extension* of the Fourier transform to allow analysis of broader class of signals and systems

- In particular, Fourier transform *cannot* handle large (and important) classes of signals and *unstable* systems, i.e. when

$$\int_{-\infty}^{\infty} |x(t)| dt = \infty$$

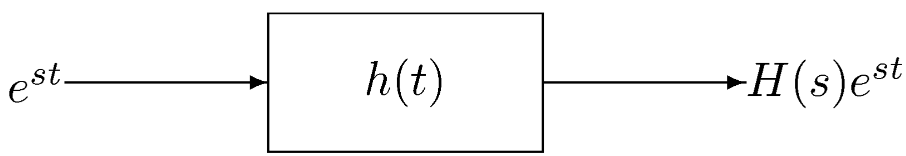
Motivation for the Laplace Transform (continued)

- In many applications, we do need to deal with *unstable* systems, e.g.
  - — Stabilizing an inverted pendulum
  - — Stabilizing an airplane or space shuttle

    ⋮

  - — Instability is *desired* in some applications, e.g. oscillators and lasers

- How do we analyze such signals/systems?
  Recall from Lecture #5, eigenfunction property of LTI systems:



$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt \qquad \text{(assuming this converges)}$$

- — e^{st} is an eigenfunction of *any* LTI system
- — $s = \sigma + j\omega$ can be complex in general

The (Bilateral) Laplace Transform

$$x(t) \longleftrightarrow X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt = \mathcal{L}\{x(t)\}$$

$s = \sigma + j\omega$ is a *complex* variable – Now we explore the full range of s

Basic ideas:

absolute integrability needed

(1) $$X(s) = X(\sigma + j\omega) = \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t} dt = \mathcal{F}\{x(t) e^{-\sigma t}\}$$

(2) A critical issue in dealing with Laplace transform is convergence:

— *X*(*s*) generally exists only for *some* values of *s*, located in what is called the *region of convergence* (ROC)

$$\text{ROC} = \{s = \sigma + j\omega \text{ so that } \int_{-\infty}^{\infty} \underbrace{|x(t) e^{-\sigma t}|}_{\substack{\text{Depends} \\ \text{only on } \sigma \\ \text{not on } \omega}} dt < \infty\}$$

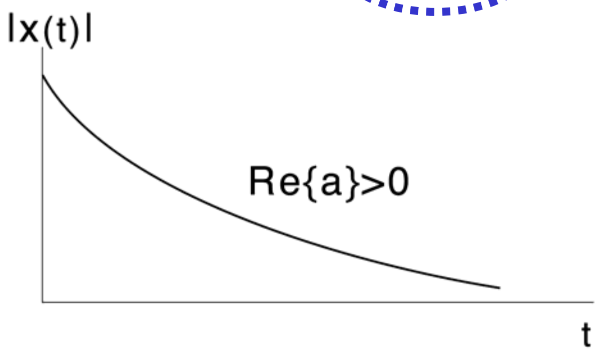
absolute integrability condition

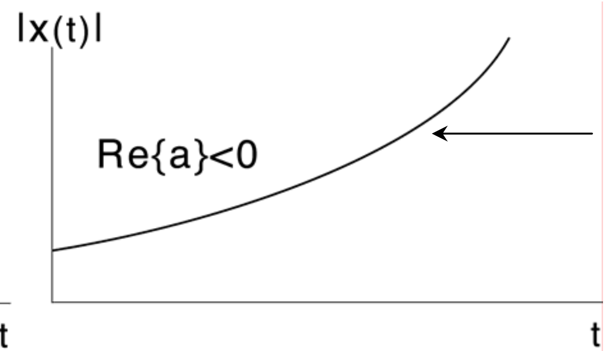
(3) If $s = j\omega$ is in the ROC (i.e. $\sigma = 0$), then

$$X(s)|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

Example #1:

$$x_1(t) = e^{-at}u(t) \qquad (a \text{ – an arbitrary real or complex number})$$





Unstable:

- no *Fourier Transform*
- but *Laplace Transform* exists

$$X_1(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt$$

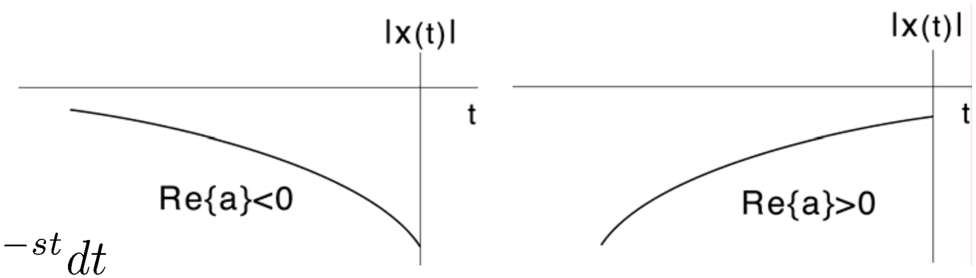
$$= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = -\frac{1}{s+a}[e^{-(s+a)\infty} - 1]$$

This converges only if $Re(s+a) > 0$, i.e. $Re(s) > -Re(a)$

$$\Downarrow$$

$$X_1(s) = \frac{1}{s+a}, \quad \underbrace{\Re e\{s\} > -\Re e\{a\}}_{\text{ROC}}$$

Example #2:



$$x_2(t) = -e^{-at}u(-t)$$

$$X_2(s) = -\int_{-\infty}^{\infty} e^{-at}u(-t)e^{-st}dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}dt$$

$$= +\frac{1}{s+a}e^{-(s+a)t}\Big|_{-\infty}^{0} = \frac{1}{s+a}[1 - e^{(s+a)\infty}]$$

This converges only if $Re(s+a) < 0$, i.e. $Re(s) < -Re(a)$

$$X_2(s) = \frac{1}{s+a}, \quad \underbrace{\Re e\{s\} < -\Re e\{a\}}_{\text{ROC}}$$ – Same as $X_1(s)$, but different ROC

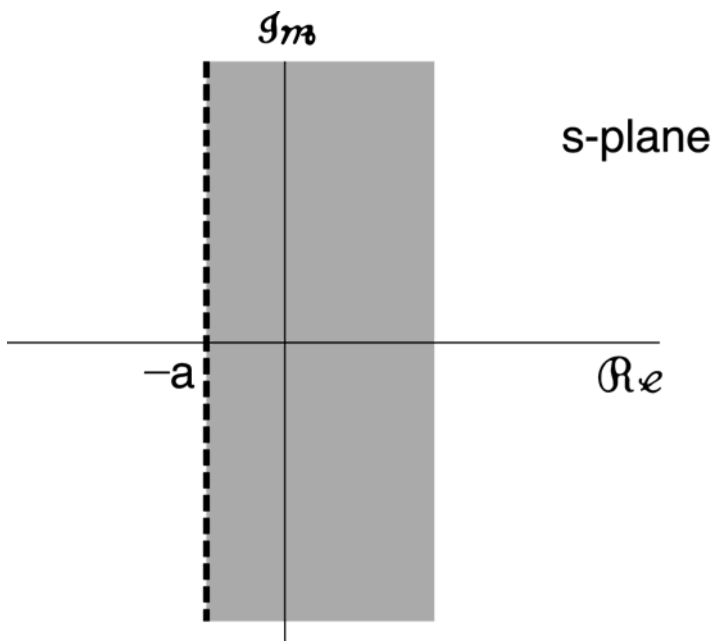
Key Point (and key difference from *FT*): Need *both* $X(s)$ *and* ROC to uniquely determine $x(t)$. No such an issue for *FT*.

Graphical Visualization of the ROC

Example #1

$$X_1(s) = \frac{1}{s+a}, \quad \Re e\{s\} > -\Re e\{a\}$$

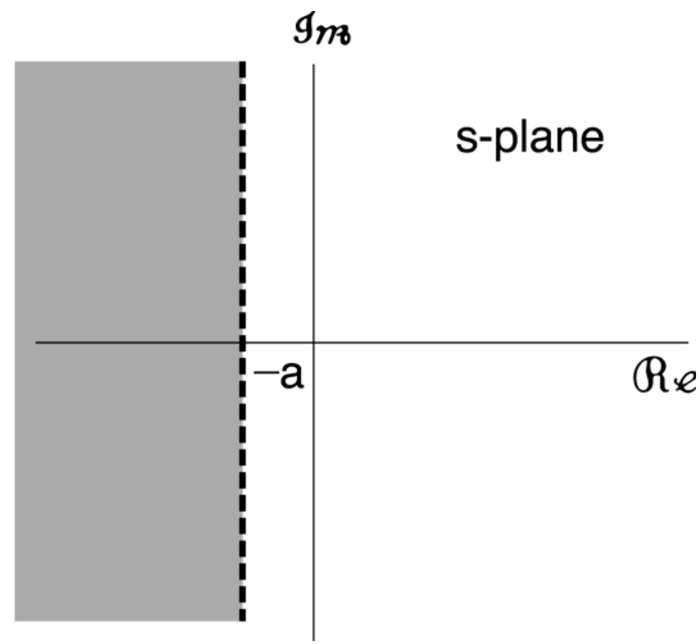
$x_1(t) = e^{-at}u(t)$ - right-sided signal



Example #2

$$X_2(s) = \frac{1}{s+a}, \quad \Re e\{s\} < -\Re e\{a\}$$

$x_2(t) = -e^{-at}u(-t)$ - left-sided signal



Rational Transforms

- Many (but by no means all) Laplace transforms of interest to us are rational functions of s (e.g., Examples #1 and #2; in general, impulse responses of LTI systems described by LCCDEs), where

$$X(s) = \frac{N(s)}{D(s)}, \quad N(s), D(s) - \text{polynomials in } s$$

- Roots of $N(s)$ = *zeros* of $X(s)$

- Roots of $D(s)$ = *poles* of $X(s)$

- Any $x(t)$ consisting of a linear combination of complex exponentials for $t > 0$ and for $t < 0$ (e.g., as in Example #1 and #2) has a rational Laplace transform.

Example #3

$$x(t) = 3e^{2t}u(t) - 2e^{-t}u(t)$$

$$X(s) = \int_0^{\infty} [3e^{2t} - 2e^{-t}]e^{-st}dt$$

$$= 3\underbrace{\int_0^{\infty} e^{-(s-2)t}dt}_{\text{Requires } \Re e\{s\} > 2} - 2\underbrace{\int_0^{\infty} e^{-(s+1)t}dt}_{\text{Requires } \Re e\{s\} > -1}$$

BOTH required → ROC intersection

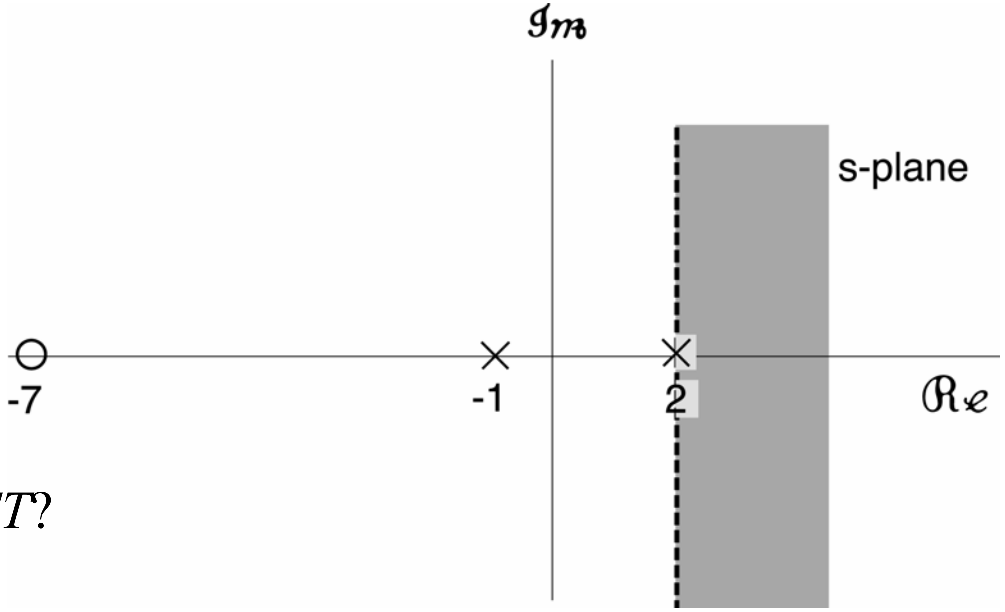
$$\Downarrow$$

$$X(s) = \frac{3}{s-2} - \frac{2}{s+1} = \frac{s+7}{(s-2)(s+1)} = \frac{s+7}{s^2-s-2} \quad \Re e\{s\} > 2$$

Notation:

× — *pole*

o — *zero*

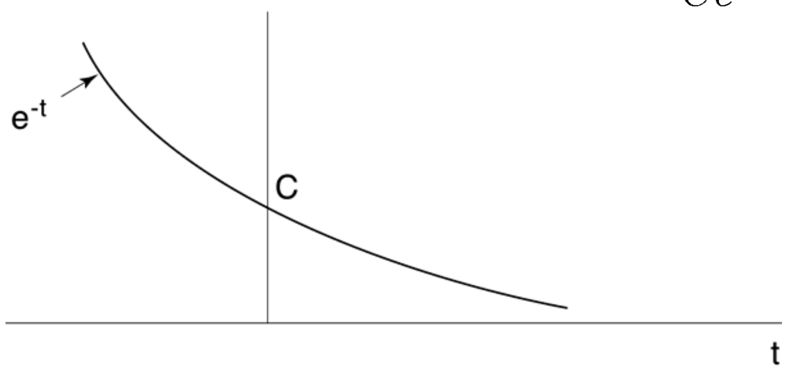


Q: Does $x(t)$ have FT?

Laplace Transforms and ROCs

- Some signals do not have Laplace Transforms (have no ROC)

(a) $x(t) = Ce^{-t}$ for all t since $\displaystyle\int_{-\infty}^{\infty} \underbrace{|x(t)e^{-\sigma t}|}_{Ce^{-(\sigma+1)t}} dt = \infty$ for all σ



(b) $x(t) = e^{j\omega_0 t}$ for all t $\quad$ *FT*: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$\int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt = \int_{-\infty}^{\infty} e^{-\sigma t}dt = \infty \text{ for all } \sigma$$

$X(s)$ is defined only in ROC; we don't allow impulses in LTs

Properties of the ROC

- The ROC can take on only a small number of different forms

1) The ROC consists of a collection of lines parallel to the $j\omega$-axis in the s-plane (i.e. the ROC only depends on σ).
   Why?

$$\int_{-\infty}^{\infty} |x(t)e^{-st}|dt = \int_{-\infty}^{\infty} |x(t)e^{-\sigma t}|dt < \infty \text{ depends only on } \sigma = \Re e\{s\}$$

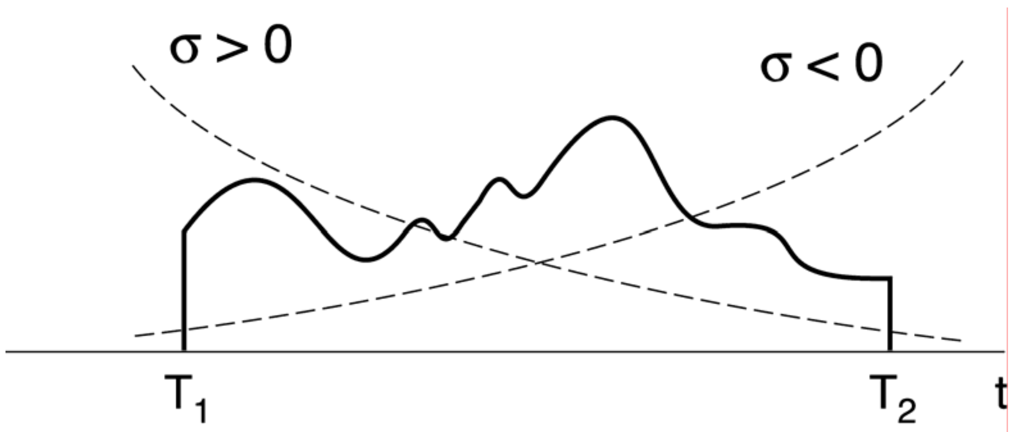
2) If $X(s)$ is rational, then the ROC does not contain any poles.
   Why?

$$\text{Poles are places where } D(s) = 0$$

$$\Rightarrow \; X(s) = \frac{N(s)}{D(s)} = \infty \quad \text{Not convergent.}$$

More Properties

3) If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

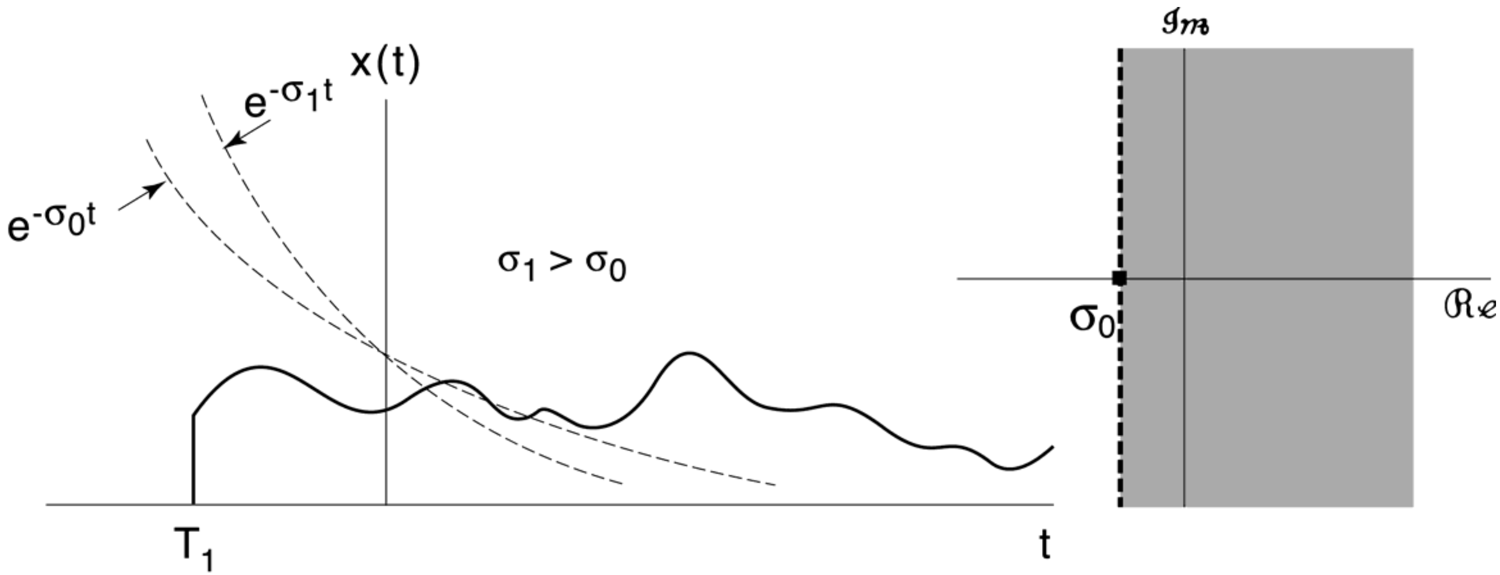


$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \underbrace{\int_{T_1}^{T_2} x(t)e^{-st}dt}_{\text{A finite integration interval}}$$

$$< \infty \quad \text{if} \int_{T_1}^{T_2} |x(t)|dt < \infty$$

ROC Properties that Depend on Which Side You Are On - I

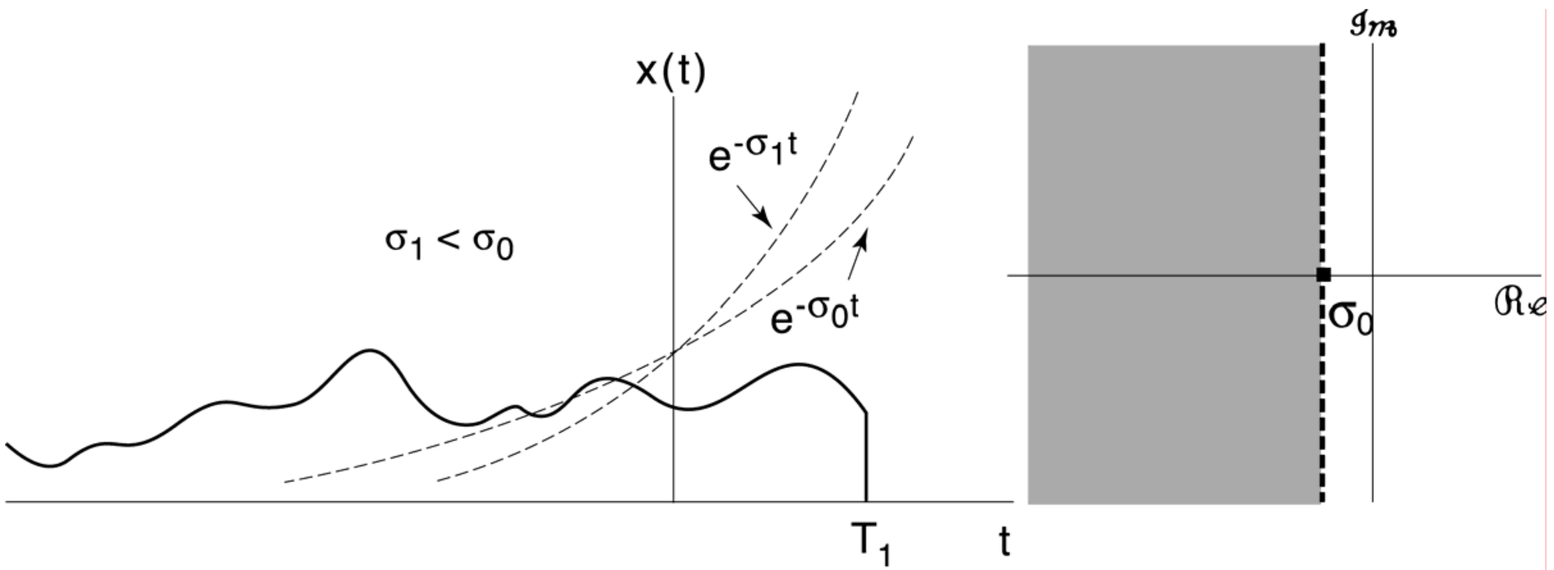
4) If $x(t)$ is right-sided (i.e. if it is zero *before* some time), and if $Re(s) = \sigma_o$ is in the ROC, then all values of s for which $Re(s) > \sigma_o$ are also in the ROC.



ROC is a right half plane (RHP)

ROC Properties that Depend on Which Side You Are On - II

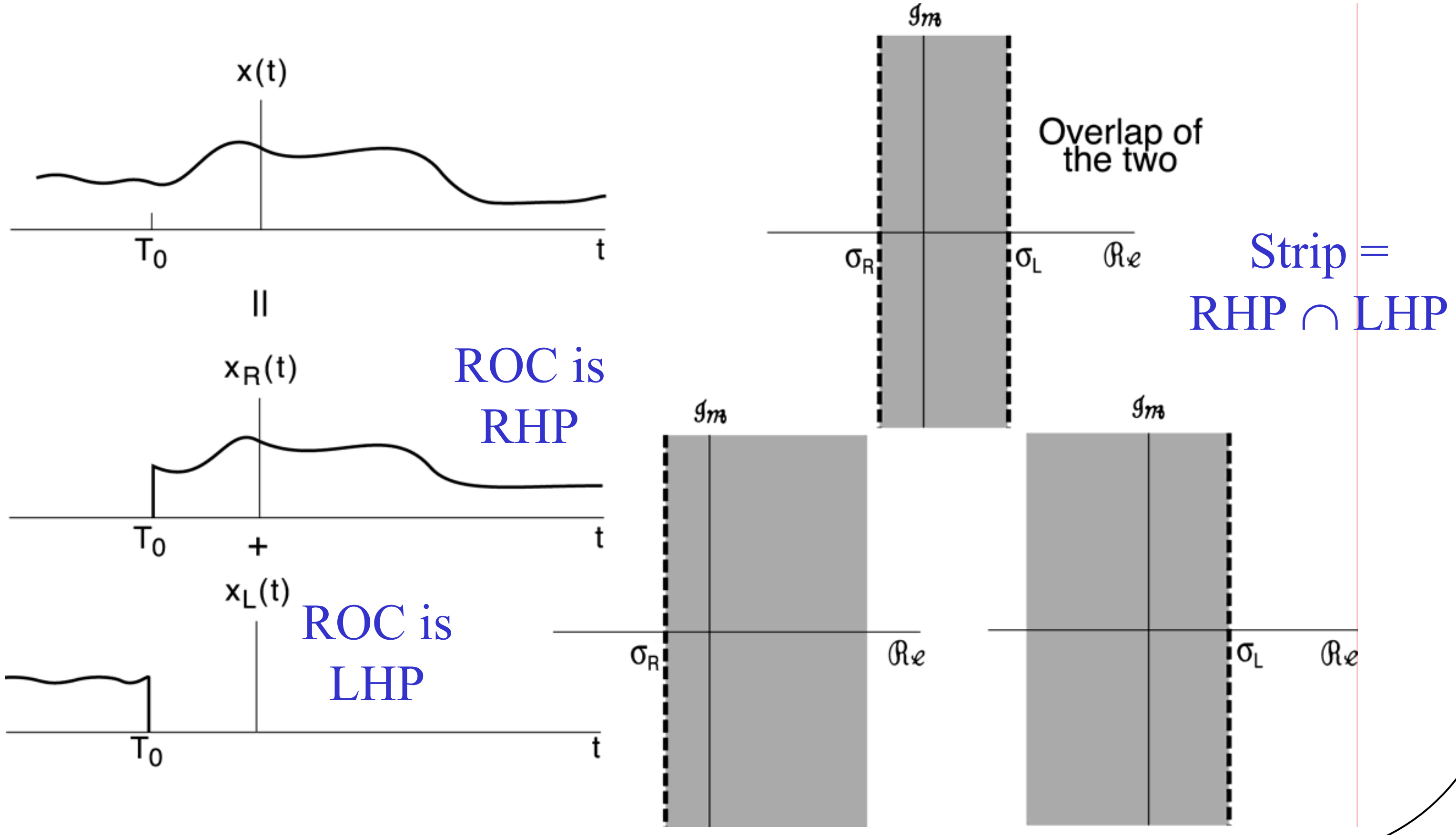
5) If $x(t)$ is left-sided (i.e. if it is zero *after* some time), and if $Re(s) = \sigma_o$ is in the ROC, then all values of s for which $Re(s) < \sigma_o$ are also in the ROC.



ROC is a left half plane (LHP)

Still More ROC Properties

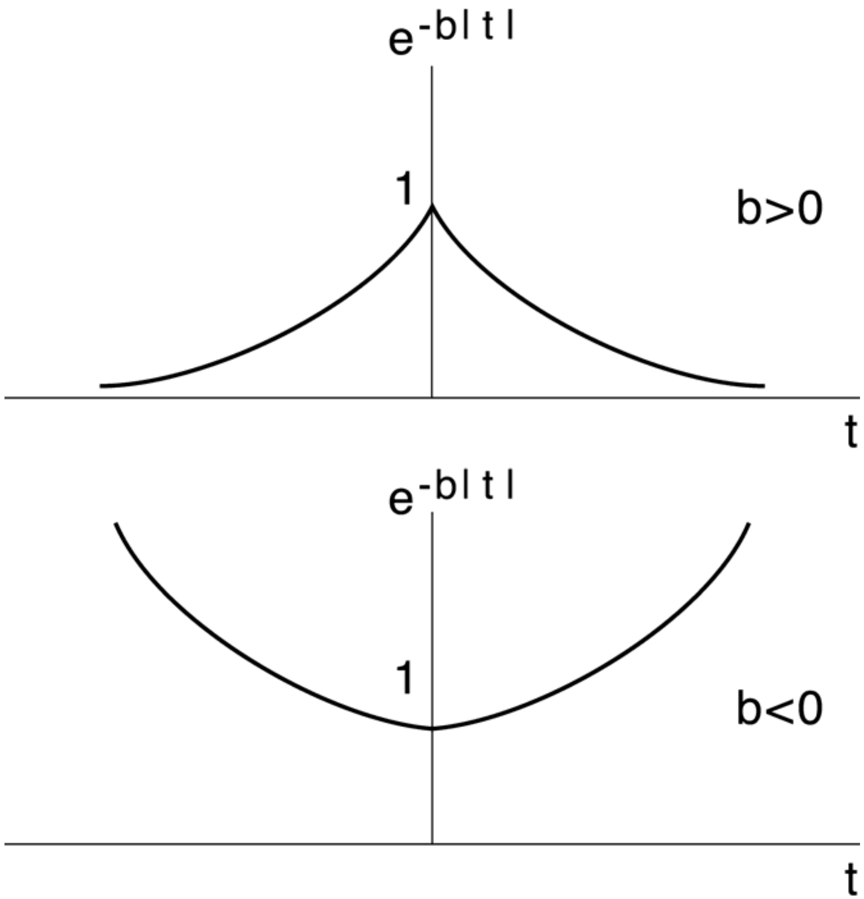
6) If $x(t)$ is two-sided and if the line $Re(s) = \sigma_o$ is in the ROC, then the ROC consists of a strip in the s-plane that includes the line $Re(s) = \sigma_o$.



Example:

$$x(t) = e^{-b|t|}$$

Intuition?



- Okay: multiply by constant (e^{0t}) and will be integrable

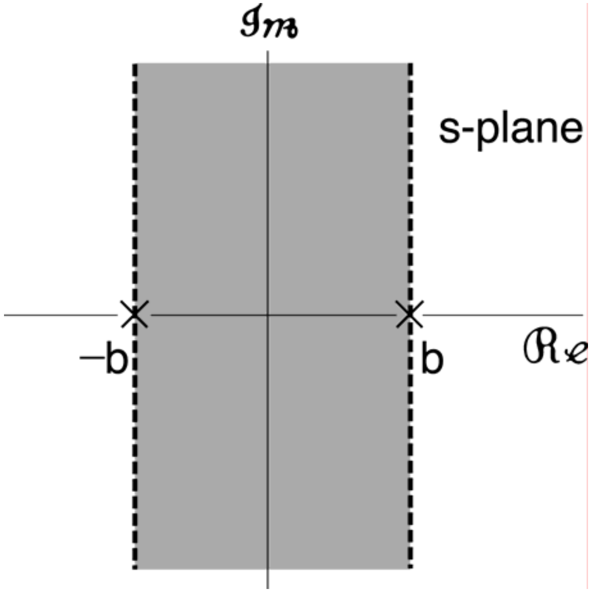
- Looks bad: no $e^{\sigma t}$ will dampen both sides

Example (continued):

$$x(t) = e^{bt}u(-t) + e^{-bt}u(t)$$

$$e^{bt}u(-t) \Downarrow -\frac{1}{s-b}, \Re e\{s\} < b \qquad e^{-bt}u(t) \Downarrow \frac{1}{s+b}, \Re e\{s\} > -b$$

Overlap if $b > 0 \Rightarrow X(s) = \frac{-2b}{s^2-b^2}$, with ROC:

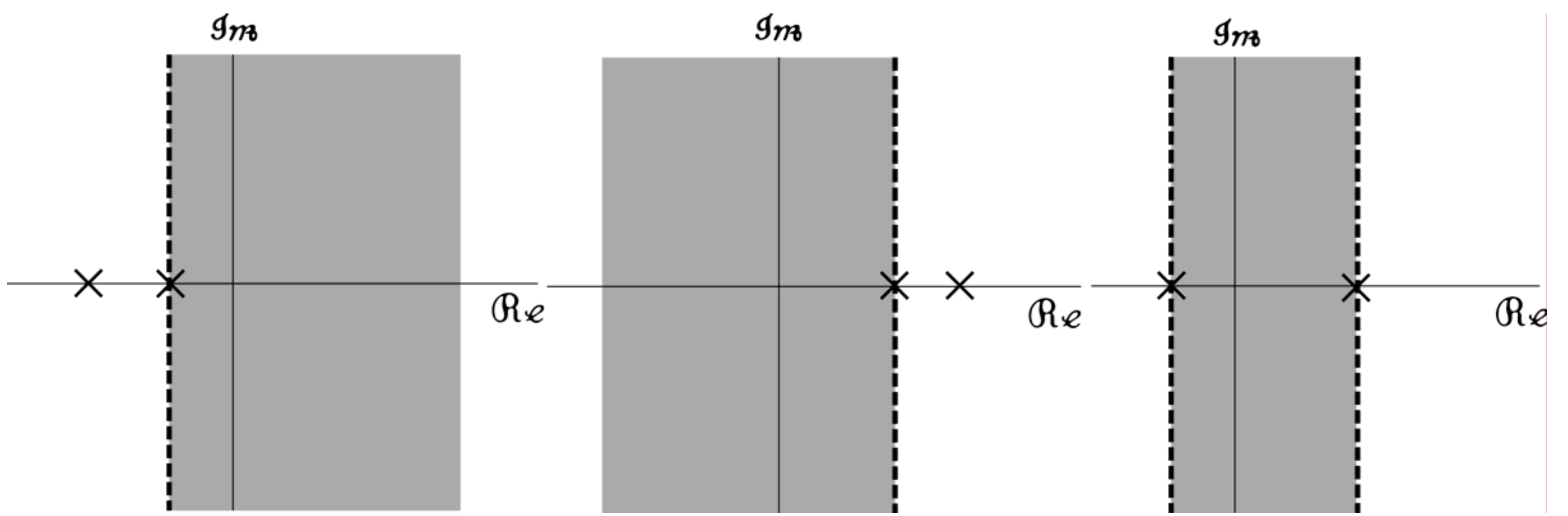


What if $b < 0$? ⇒ No overlap ⇒ No Laplace Transform

Properties, Properties

7) If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

8) Suppose $X(s)$ is rational, then
   (a) If $x(t)$ is right-sided, the ROC is to the right of the rightmost pole.
   (b) If $x(t)$ is left-sided, the ROC is to the left of the leftmost pole.



(a) (b) (c)

9) If ROC of $X(s)$ includes the $j\omega$-axis, then *FT* of $x(t)$ exists.

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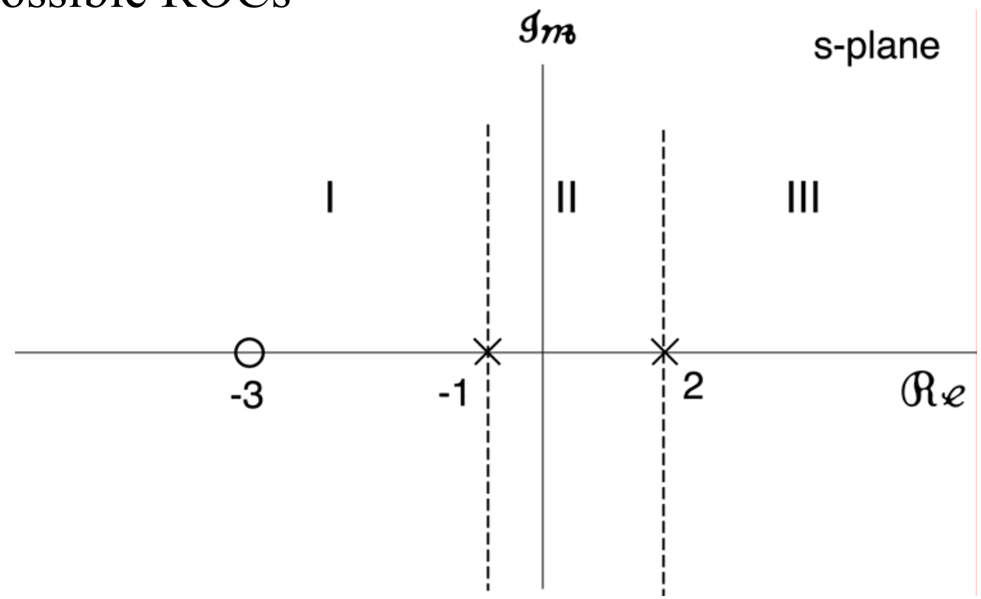
Example:

$$X(s) = \frac{(s+3)}{(s+1)(s-2)}$$

Three possible ROCs



			Fourier Transform exists?
$x(t)$ is right-sided	ROC:	III	No
$x(t)$ is left-sided	ROC:	I	No
$x(t)$ extends for all time	ROC:	II	Yes